第九周作业答案

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习题 1 (Stein,Ch2,T14) In Exercise 6 of the previous chapter we saw that $m(B) = v_d r^d$, whenever B is a ball of radius r in $\mathbb{R}^d$ and $v_d = m(B_1)$, with B_1 the unit ball. Here we evaluate the constant v_d.

(a) For $d = 2$, prove using Corollary 3.8 that

$$v_2 = 2\int_{-1}^{1} (1-x^2)^{1/2}\,\mathrm{d}x$$

and hence by elementary calculus, that $v_2 = \pi$.

(b) By similar methods, show that

$$v_d = 2v_{d-1}\int_{0}^{1} (1-x^2)^{(d-1)/2}\,\mathrm{d}x$$

(c) The result is

$$v_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

Another derivation is in Exercise 5 in Chapter 6 below. Relevant facts about the gamma and beta functions can be found in Chapter 6 of Book II.

证明 (1). 设

$$f(x) = \begin{cases} (1-x^2)^{\frac{1}{2}}, & x \in [-1,1] \\ 0, & \text{else} \end{cases}$$

记 $A = \{(x,y) : -1 \le x \le 1, 0 \le y \le f(x)\}$，则由课本推论 3.8，$A$ 可测，且

$$\begin{aligned} v_2 = 2m(A) &= 2\int_{\mathbb{R}^2} \chi_A \mathrm{d}x\mathrm{d}y \\ &= 2\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_A \mathrm{d}y \right) \mathrm{d}x = 2\int_{\mathbb{R}} \int_0^{f(x)} 1\mathrm{d}y\mathrm{d}x \\ &= 2\int_{-1}^{1} f(x)\mathrm{d}x = \pi \end{aligned}$$

(2). 定义 $f : \mathbb{R}^{d-1} \to \mathbb{R}$ 如下

$$f(x) = \begin{cases} (1-|x|^2)^{\frac{1}{2}}, & 0 \le |x| \le 1 \\ 0, & \text{else} \end{cases}$$

记 $A = \{(x,y) : x \in \mathbb{R}^{d-1}, y \in \mathbb{R}, 0 \le |x| \le 1, 0 \le y \le f(x)\}$，当固定 y 时，$A^y = \{x : 0 \le |x| \le (1-y^2)^{\frac{1}{2}}\}$，

而由定义

$$m(A^y) = v_{d-1} \cdot (1-y^2)^{\frac{d-1}{2}}$$

所以

$$\begin{aligned} v_d = 2m(A) &= 2\int_{\mathbb{R}^d} \chi_A \mathrm{d}x\mathrm{d}y \\ &= 2\int_0^1 \int_{\mathbb{R}^{d-1}} \chi_A(x,y)\mathrm{d}x\mathrm{d}y \\ &= 2\int_0^1 m(A^y)\mathrm{d}y = 2v_{d-1}\int_0^1 (1-y^2)^{\frac{d-1}{2}}\mathrm{d}y \end{aligned}$$

(3). 设 $y=\sqrt{x}$，则 $\mathrm{d}y = \frac{1}{2\sqrt{x}}\mathrm{d}x$，因此

$$\begin{aligned} \int_0^1 (1-y^2)^{\frac{d-1}{2}}\mathrm{d}y = \frac{1}{2}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{d-1}{2}}\mathrm{d}x &= \frac{1}{2}B\left(\frac{1}{2}, \frac{d+1}{2}\right) \\ &= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)} \end{aligned}$$

因此

$$\begin{aligned} v_d = \sqrt{\pi}v_{d-1}\cdot\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)} &= \left(\sqrt{\pi}\right)^2 v_{d-2}\cdot\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\cdot\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \\ &= \left(\sqrt{\pi}\right)^{d-2} v_2\cdot\frac{\Gamma(2)}{\Gamma\left(\frac{d}{2}+1\right)} = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \end{aligned}$$

□

习题 2 (Stein,Ch2,T17) Suppose f is defined on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} a_n, & n \le x < n+1,\ n \le y < n+1,\ n \ge 0 \\ -a_n, & n \le x < n+1,\ n+1 \le y < n+2,\ n \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

Here $a_n = \sum_{k\le n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

(a) Verify that each slice f^y and f_x is integrable. Also for all x, $\int f_x(y)\,dy = 0$, and hence

$$\int\left(\int f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = 0$$

.

(b) However, $\int f^y(x)\,\mathrm{d}x = a_0$ if $0 \le y < 1$, and $\int f^y(x)\,\mathrm{d}x = a_n - a_{n-1}$ if $n \le y < n+1$ with $n \ge 1$. Hence $y \mapsto \int f^y(x)\,\mathrm{d}x$ is integrable on $(0,\infty)$ and

$$\int\left(\int f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = s$$

(c) Note that

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y = \infty$$

证明 (a). 固定 $x\in\mathbb{R}$，若 $x\in(-\infty,0)$，则 $f_x(y)=f(x,y)=0$，所以 $f_x\in L^1(\mathbb{R})$，且

$$\int_{\mathbb{R}} f_x(y)\,\mathrm{d}y=0$$

若 $x\in[n,n+1)$，其中 $n\geq 0$，则

$$f_x(y)=\begin{cases} a_n, & n\leq y<n+1\\ -a_n, & n+1\leq y<n+2\\ 0, & \text{otherwise}\end{cases}$$

于是

$$\int_{\mathbb{R}} |f_x(y)|\,\mathrm{d}y=a_n+a_n=2a_n<\infty$$

所以 $f_x\in L^1(\mathbb{R})$，同时

$$\int_{\mathbb{R}} f_x(y)\,dy=a_n-a_n=0.$$

故

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x=\int_{\mathbb{R}} 0\,\mathrm{d}x=0$$

再固定 $y\in\mathbb{R}$，若 $y<0$，则 $f^y(x)=0$，所以 $f^y\in L^1(\mathbb{R})$。若 $0\leq y<1$，则

$$f^y(x)=\begin{cases} a_0, & 0\leq x<1\\ 0, & \text{otherwise}\end{cases}$$

因此 $f^y\in L^1(\mathbb{R})$。若 $n\leq y<n+1$，其中 $n\geq 1$，则

$$f^y(x)=\begin{cases} -a_{n-1}, & n-1\leq x<n\\ a_n, & n\leq x<n+1\\ 0, & \text{otherwise}\end{cases}$$

于是

$$\int_{\mathbb{R}} |f^y(x)|\,\mathrm{d}x=a_{n-1}+a_n<\infty$$

因此每个切片 $f^y\in L^1(\mathbb{R})$

(b). 若 $0\leq y<1$，则

$$\int_{\mathbb{R}} f^y(x)\,\mathrm{d}x=a_0=b_0$$

若 $n\leq y<n+1$，其中 $n\geq 1$，则

$$\int_{\mathbb{R}} f^y(x)\,\mathrm{d}x=\int_{n-1}^{n}(-a_{n-1})\,\mathrm{d}x+\int_{n}^{n+1} a_n\,\mathrm{d}x=a_n-a_{n-1}=b_n$$

再设

$$g(y)=\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x$$

则由上分析知

$$g(y)=\begin{cases} b_n, & n\leq y<n+1,\ n\geq 0\\ 0, & y<0\end{cases}$$

于是 $g \geq 0$，并且

$$\int_{\mathbb{R}} |g(y)|\,\mathrm{d}y = \int_0^\infty g(y)\,\mathrm{d}y = \sum_{n=0}^\infty \int_n^{n+1} b_n\,\mathrm{d}y = \sum_{n=0}^\infty b_n = s < \infty$$

所以 $g \in L^1(0,\infty)$，并且

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = s$$

(c). 设 $Q_n^+ = [n,n+1)\times[n,n+1), Q_n^- = [n,n+1)\times[n+1,n+2)$，则在每个 Q_n^+, Q_n^- 上有 $|f| = a_n$，因此

$$\int_{\mathbb{R}^2} |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y = \sum_{n=0}^\infty \left(\int_{Q_n^+} a_n\,\mathrm{d}x\,\mathrm{d}y + \int_{Q_n^-} a_n\,\mathrm{d}x\,\mathrm{d}y \right) = 2\sum_{n=0}^\infty a_n \geq 2\sum_{n=0}^\infty b_0 = \infty$$

□

注 这题告诉我们 Tonelli 定理中的非负是必要的

习题 3 (Stein,Ch2,T20) The problem (highlighted in the discussion preceding Fubini's theorem) that certain slices of measurable sets can be non-measurable may be avoided by restricting attention to Borel measurable functions and Borel sets. In fact, prove the following:

Suppose E is a Borel set in $\mathbb{R}^2$. Then for every y, the slice E^y is a Borel set in $\mathbb{R}$.

Hint: Consider the collection $\mathcal{C}$ of subsets E of $\mathbb{R}^2$ with the property that each slice E^y is a Borel set in $\mathbb{R}$. Verify that $\mathcal{C}$ is a σ-algebra that contains the open sets.

证明 设 $E^y = \{x \in \mathbb{R} : (x,y) \in E\}$，令

$$\mathcal{C} = \{E \subset \mathbb{R}^2 : 对任意y \in \mathbb{R},\ E^y是\mathbb{R}中的\ \text{Borel}\ 集\}$$

下证明 $\mathcal{C}$ 是一个 σ-代数

(1) 因为对任意 $y \in \mathbb{R}$，有 $(\mathbb{R}^2)^y = \mathbb{R}$，而 $\mathbb{R}$ 是 *Borel* 集，所以 $\mathbb{R}^2 \in \mathcal{C}$，显然 $\varnothing \in \mathcal{C}$

(2) 若 $E \in \mathcal{C}$，固定 $y \in \mathbb{R}$，对任意 $x \in \mathbb{R}$，有

$$x \in (E^c)^y \iff (x,y) \in E^c \iff (x,y) \notin E \iff x \notin E^y \iff x \in (E^y)^c$$

因此 $(E^c)^y = (E^y)^c$，故 $(E^c)^y$ 也是 Borel 集，$E^c \in \mathcal{C}$

(3) 若 $E_n \in \mathcal{C}, n = 1,2,\ldots$，则对任意 $y \in \mathbb{R}$，有

$$\left(\bigcup_{n=1}^\infty E_n \right)^y = \{x \in \mathbb{R} : (x,y) \in \bigcup_{n=1}^\infty E_n\} = \bigcup_{n=1}^\infty E_n^y$$

由于每个 E_n^y 都是 Borel 集，且 Borel 集对可数并封闭，故 $(\bigcup_{n=1}^\infty E_n)^y$ 是 Borel 集，所以 $\bigcup_{n=1}^\infty E_n \in \mathcal{C}$

所以 $\mathcal{C}$ 是一个 σ-代数，下面证明所有开集都属于 $\mathcal{C}$，设 $U \subset \mathbb{R}^2$ 是开集，固定 $y \in \mathbb{R}$。若 $x \in U^y$，则 $(x,y) \in U$，由 U 开知，存在 $\varepsilon > 0$，使得

$$B((x,y),\varepsilon) \subset U$$

于是当 $|x' - x| < \varepsilon$ 时，$(x',y) \in U$，即 $x' \in U^y$。所以 U^y 是 $\mathbb{R}$ 中的开集，从而是 Borel 集，$U \in \mathcal{C}$，故 $\mathcal{C}$ 是包含 $\mathbb{R}^2$ 中所有开集的 σ-代数，故

$$\mathcal{B}(\mathbb{R}^2) \subset \mathcal{C}$$

所以若 E 是 $\mathbb{R}^2$ 中的 Borel 集，则 $E\in\mathcal{C}$，即对任意 $y\in\mathbb{R}$，切片 E^y 是 $\mathbb{R}$ 中的 Borel 集 □

习题 4 Prove that $\forall f\in L^p, 1\leq p<\infty$

$$||f||_p^p = p\int_0^\infty m(\{|f|>t\})t^{p-1}\mathrm{d}t$$

证明 对每个 x，由微积分知识知

$$|f(x)|^p=\int_0^{|f(x)|} pt^{p-1}\,\mathrm{d}t=\int_0^\infty pt^{p-1}\chi_{\{t<|f(x)|\}}\,\mathrm{d}t$$

由于被积函数 $pt^{p-1}\chi_{\{t<|f(x)|\}}\geq 0$ 且可测，由 Tonelli 定理

$$\|f\|_p^p=\int_{\mathbb{R}^d}|f(x)|^p\,\mathrm{d}m=\int_{\mathbb{R}^d}\left(\int_0^\infty pt^{p-1}\chi_{\{t<|f(x)|\}}\,\mathrm{d}t\right)\mathrm{d}m=\int_0^\infty pt^{p-1}\left(\int\chi_{\{|f(x)|>t\}}\,\mathrm{d}m\right)\mathrm{d}t$$

注意到

$$\int\chi_{\{|f(x)|>t\}}\,\mathrm{d}m=m(\{|f|>t\})$$

因此

$$\|f\|_p^p=p\int_0^\infty m(\{|f|>t\})t^{p-1}\,\mathrm{d}t$$

□