Real Analysis: Chapter 3

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A thought

Problem 1

Addition Problem 1: Measurable in the sense of Lebsgue $\Leftrightarrow$ measurable in the sense of Caratheodory.

Proof. (a) Suppose E is Lebsgue-measurable. Let $\epsilon > 0$. Choose

$$F(closed) \subset E \subset U(open) \text{ s.t. } m(U \setminus F) < \epsilon.$$

Suppose $A \subset \mathbb{R}^k$ and V be an **open set** containing A. Then $(A \setminus E) \subset (V \setminus F)$ and $(A \bigcap E) \subset (V \bigcap U)$, which follows that

$$\begin{aligned} m^*(A \setminus E) + m^*(A \cap E) &\leq m(V \setminus F) + m(V \cap U) \\ &\leq m(V \setminus U) + m(U \setminus F) + m(V \cap U) \\ &\leq m(V) + \epsilon. \end{aligned}$$

Since ϵ and V is arbitrary, E is Caratheodory-measurable.
(b) Suppose E is Caratheodory-measurable. The case when $m^*(E) < \infty$ is easy as follows:
$\exists E \subset U(open)$ s.t. $m(U) < m^*(E) + \epsilon$. Then

$$m(U) = m^*(U \cup E) + m^*(U \setminus E) = m^*(E) + m^*(U \setminus E) \implies m^*(U \setminus E) < \epsilon.$$

when $m^*(E) = \infty$, just consider $E \bigcap B_N(0)$. $\square$

Exercises in Stein Chapter 3

Exercise 1

(a) To prove the first two properties of good kernels, it's sufficient that

$$\int_{\mathbb{R}^d} K_\delta(x)dx = \int_{\mathbb{R}^d} \frac{1}{\delta^d}\varphi(\frac{x}{\delta})dx = \int_{\mathbb{R}^d} \varphi(x)dx = 1 \text{ and } \int_{\mathbb{R}^d} |K_\delta(x)| = \int_{\mathbb{R}^d} |\varphi(x)| < \infty$$

.
For the last, it's sufficient that for every $\eta > 0$:

$$\begin{aligned} \int_{|x|\geq\eta} |K_\delta(x)| &= \int_{\mathbb{R}^d} |K_\delta(x)|\chi_{\{x:|x|\geq\eta\}}(x) = \int_{\mathbb{R}^d} |\varphi(x)|\chi_{\{x:|x|\geq\eta\}}(\delta x)dx \\ &= \int_{\mathbb{R}^d} |\varphi(x)|\chi_{\{x:|x|\geq\frac{\eta}{\delta}\}}(x) \to 0 \ (\ \delta \to 0\ ) \end{aligned}$$

(b) Suppose $|\varphi| < M$ and $E := supp(\varphi) \subset B_r(o)$ where r is sufficiently large. Therefore we have

$$|K_\delta(x)| \leq \frac{1}{\delta^d}|\varphi(\frac{x}{\delta})| \leq \frac{M}{\delta^d}$$

and when $|\frac{x}{\delta}| \leq r$:

$$|K_\delta(x)| \leq \frac{1}{\delta^d}|\varphi(\frac{x}{\delta})| \leq \frac{M\delta}{\delta^{d+1}} \leq \frac{Mr^{d+1}\delta}{|x|^{d+1}}.$$

When $|\frac{x}{\delta}| \leq r$, it's equal to 0.

(c) Assume $\epsilon > 0$. Then for such ϵ, there exist $\eta > 0$ s.t.

$$|h| < \eta \implies \|f(x-h) - f(x)\|_{L^1(\mathbb{R}^d)} < \frac{\epsilon}{2A}$$

If we choose δ small enough s.t. $\int_{|x|\geq\eta} |K_\delta(x)| < \epsilon/2$, then

$$\begin{aligned}
\|(f * K_\delta) - f\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)K_\delta(y)dy - \int_{\mathbb{R}^d} f(x)K_\delta(y)dy \right| dx \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)| K_\delta(y)dy \right) dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)| K_\delta(y)dx \right) dy \\
&= \int_{\mathbb{R}^d} K_\delta(y) \|f(x-y) - f(y)\|_{L^1(\mathbb{R}^d)} dy \\
&= \int_{|y|<\eta} K_\delta(y) \|f(x-y) - f(y)\|_{L^1(\mathbb{R}^d)} dy + \int_{|y|\geq\eta} K_\delta(y) \|f(x-y) - f(y)\|_{L^1(\mathbb{R}^d)} dy \\
&< A \cdot \frac{\epsilon}{2A} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Exercise 2

Proof. Suppose G_δ is an approximation to the identity, then it's clear $K_\delta + G_\delta$ is an approximation to the identity. Then

$$(f * K_\delta + f * G_\delta)(x) = (f * (K_\delta + G_\delta))(x) \to f(x) \text{ a.e. x.}$$

Moveover,

$$(f * G_\delta)(x) \to f(x) \text{ a.e. x,}$$

which immediatelt follows that

$$(f * K_\delta)(x) \to 0 \text{ a.e. x.}$$

□

Exercise 3

Proof. (a) Define $B_r(0) = (-r, r)$. Since 0 is a point of lebesgue density of the set $E \subset \mathbb{R}$,

$$\lim_{r\to 0} \frac{m(B_r(0) \cap E)}{m(B_r(0))} = 1 \implies \lim_{r\to 0} \frac{m(B_r(0) \cap (-E))}{m(B_r(0))} = 1$$

which implies that

$$\exists r_0 > 0, \forall r < r_0 \left(m(B_r(0) \cap E) > \frac{9}{10} \cdot 2r = \frac{9r}{5} \text{ and } m(B_r(0) \cap (-E)) > \frac{9}{10} \cdot 2r = \frac{9r}{5} \right).$$

It follows that

$$m\left(\left(B_r(0) \cap E \right) \bigcap \left(B_r(0) \cap (-E) \right) \right) > 0$$

Otherwise,

$$2r = m(B_r(0)) \leq m\left(\left(B_r(0) \cap E \right) \bigcap \left(B_r(0) \cap (-E) \right) \right) = m\left(B_r(0) \cap E \right) + m\left(B_r(0) \cap (-E) \right) = \frac{18}{5} r.$$

A contradiction!
Therefore, $\exists\ x_r > 0$ s.t. $x_r \in B_r(0)\bigcap E$ and exist the corrsponding $-x_R$ s.t. $x_r \in B_r(0)\bigcap E$. Since r is arbitrary, there is an infinite sequence of points $x_n \in E$ subject to the condition.

(b) Similar to proof of (a). Since 0 is a point of lebesgue density of the set $E \subset \mathbb{R}$,

$$\begin{aligned}
&\lim_{r\to 0}\frac{m(B_r(0)\cap E)}{m(B_r(0))} = 1\\
\implies &\lim_{r\to 0}\frac{m(B_r(0)\cap 2E)}{m(B_r(0))} = \lim_{2r\to 0}\frac{m(B_{2r}(0)\cap 2E)}{m(B_{2r}(0))} \geq \lim_{r\to 0}\frac{m(2(B_r(0)\cap E))}{m(B_{2r(0)})} = \lim_{r\to 0}\frac{m(B_r(0)\cap E)}{m(B_{r(0)})} = 1
\end{aligned}$$

Then the same goes for proof of (b). $\square$

Exercise 4

Proof. 1) Define $E_n = \{x \in \mathbb{R}^d : |f(x)| > \frac{1}{n}\}$. Since f is not identically zero, there exist $n \in N^*$ such that $m(E_n) > 0$.
Then E_n contains a point of lebesgue density denoted by x_0. Therefore

$$\lim_{B_r(x_0)\to 0}\frac{m(B_r(x_0)\cap E_n)}{m(B_r(x_0))} = 1 \implies \exists\ r_0 > 0\ \Big(m(B_{r_0}(x_0)\cap E_n) > 0\Big),$$

which implies that

$$\int_{B_{r_0}(x_0)}|f| \geq \int_{B_{r_0}(x_0)\cap E_n}|f| \geq \frac{m(B_{r_0}(x_0)\cap E_n)}{n}.$$

In other words there exist $\epsilon > 0$ such that $\int_{B_{r_0}(x)}|f| > \epsilon$.
Then for $|x| \geq 1$, consider $B(x) := B_{|x-x_0|+r_0}(x)$. Suppose for some $A > 0$:

$$|x| \geq 1 \implies (|x - x_0| + r)^d \leq (|x| + |x_0| + r)^d \leq \frac{1}{A}|x|^d$$

.
Then

$$f^*(x) \geq \frac{\int_{B(x)}|f|}{m(B(x))} \geq A\cdot C(d)\cdot\frac{\int_{B_{r_0(x_0)}}|f|}{|x|^d} \geq \frac{A\cdot C(d)\cdot\epsilon}{|x|^d}.$$

2) Since

$$\int_{\mathbb{R}^d}\frac{1}{|x|^d} = \int_{\mathbb{R}^d}\left(\int_R \chi_{\{f\geq\eta\}}d\eta\right)dx = \int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\chi_{\{f\geq\eta\}}dx\right)d\eta = \int_{\mathbb{R}}\frac{C(d)}{\eta}d\eta = \infty,$$

$\frac{1}{|x|^d}$ is not integrable on $\mathbb{R}^d$, which immediately follows that f^* which beats $\frac{1}{|x|^d}$ is not integrable on $\mathbb{R}^d$.
3) Since

$$\{x \in \mathbb{R}^d : 1 \leq |x| < (\frac{c}{\alpha})^{1/d}\} = \{x \in \mathbb{R}^d : |x| \geq 1 \wedge (\frac{c}{|x|^d} > \alpha)\} \subset \{x : f^*(x) > \alpha\},$$

$$m\{x : f^*(x) > \alpha\} \geq m(\{x \in \mathbb{R}^d : 1 \leq |x| < (\frac{c}{\alpha})^{1/d}\}) = vol(B_1(0))\cdot(\frac{c}{\alpha} - 1),$$

which immediately follows the conclusion. $\square$

Exercise 5

Proof. (a) We have

$$\int_{\mathbb{R}} |f| = 2\int_0^{\frac{1}{2}} \frac{dx}{x(log\frac{1}{x})^2} = \frac{2}{log(2)} < \infty.$$

(b) Suppose $|x| \le \frac{1}{2}$. WLOG, let $x > 0$. Consider the ball $B := B_{|x|}(x)$. Then

$$f^*(x) \ge \frac{1}{m(B)}\int_B |f| \ge \frac{\int_0^x |f(x)|dx}{2|x|} \ge \frac{c}{|x|log(\frac{1}{|x|})}.$$

To show it's not locally integrable, it's sufficient that for $0 < t < \frac{1}{2}$:

$$\int_0^t \frac{c}{|x|log\frac{1}{|x|}}dx = -c \cdot log(log(\frac{1}{x}))\Big|_0^t = \infty.$$

□

Exercise 6

Proof. Consider $F(x) = \int_0^x |f(y)| - \alpha x$. Then

$$x \in E_\alpha^+ \Leftrightarrow \exists h > 0\bigg(\frac{\int_x^{x+h}|f(y)|dy}{h} > \alpha\bigg) \Leftrightarrow \exists h > 0\bigg(F(x+h) > F(x)\bigg).$$

Then by applying corollary 3.5, $E_\alpha^+ = \bigcup_k (a_k, b_k)$ with each interval disjoint and $F(a_k) = F(b_k)$ which implies

$$\int_{a_k}^{b_k} |f(y)|dy = \alpha(b_k - a_k).$$

Then

$$m(E_\alpha^+) = \sum_k m((a_k, b_k)) = \sum_k \frac{\int_{a_k}^{b_k}|f(y)|dy}{\alpha} = \frac{1}{\alpha}\int_{E_\alpha^+} |f(y)|dy.$$

□

Exercise 7

Proof. Arbitrarily choose a point $x_0 \in (0,1)$. Then we know

$$\liminf_{\substack{x_0 \in I \\ m(I)\to 0}} \frac{m(E\cap I)}{m(I)} \ge \alpha > 0.$$

We know that there exist a zero-measure set denoted by $Z \subset (0,1)$ such that

$$\forall x \in (0,1)\setminus(E\cup Z)\bigg(\lim_{\substack{x_0 \in I \\ m(I)\to 0}} \frac{m(E\cap I)}{m(I)} = 0\bigg),$$

which implies that $x_0 \notin (0,1)\setminus(E\cup Z)$. Since $x_0 \in (0,1)$ is arbitrarily choosen, $(0,1) \subset (E\cup Z)$. Then $1 \ge m(E) \ge m(E\cup Z) - m(Z) = 1 \implies m(E) = 1$. □

Exercise 8

Proof. First we prove the case in $C_N := [-N, N] \subset \mathbb{R}$. A more general case can be proved as a consequence.

For $\epsilon, N > 0$, there exist a closed interval I_N with length $l_N < N$ such that

$$m(A \cap I_N) \geq (1 - \frac{\epsilon}{4N})m(I_N) \implies m(I_N \setminus A) \leq \frac{l_N}{4N}\epsilon.$$

Since $C_N \subset \bigcup_{k\in K_N}(I_N + kl_N) \subset 2C_N$ where K_N is a **finite index set**. Then consider $\bigcup_{k\in K_N}(A + k \cdot l_k)$, let $|K_N|$ denote the number of the set K_N:

$$\begin{aligned}
2N \leq |K_N| l_N \leq 4N \implies \quad & m\Big(C_N \setminus (\bigcup_{k\in K_N}(A + kl_N))\Big) \\
\leq & m\Big((\bigcup_{k\in K_N}(I_N + kl_N)) \setminus (\bigcup_{k\in K_N}(A + kl_N))\Big) \\
\leq & m\Big((\bigcup_{k\in K_N}(I_N \setminus A + kl_N))\Big) \\
= & |K_N| \cdot \frac{l_N}{4N} \cdot \epsilon \leq \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $m\Big(C_N \setminus (\bigcup_{k\in K_N}(A + kl_N))\Big) = 0$
Therefore for general case: since $\mathbb{R} = \bigcup_{N=1}^{\infty} C_N$,

$$m(\bigcup_{N=1}^{\infty} C_N / \bigcup_{n=1}^{\infty}\bigcup_{k\in K_n}(A + kl_n)) \leq m\Big(\bigcup_{N=1}^{\infty}(C_N \setminus \bigcup_{k\in K_N}(A + kl_N))\Big) = \sum_{N=1}^{\infty} m\Big(C_N \setminus \bigcup_{k\in K_N}(A + kl_N)\Big) = 0$$

□

Exercise 9

Solution 1: A few facts will help:

Fact 1: $|\delta(x) - \delta(y)| \leq |x - y|$.
Fact 2: $\delta(x)$ has BV on $[a, b]$ and then is differentiable a.e. x.
Fact 3: Every point of F is a local minimum of $\delta(x)$.

Solution 2: It suffices to show that the proposition holds for x being a point of density of F. Suppose

$$\exists 0 < \epsilon_0 < 1, \exists\{y_k\} \text{ s.t. } |y_k| \to 0\Big(\delta(x + y_k) \geq \epsilon_0|y_k|\Big).$$

Then
$$\frac{m(F \cap [x - 2|y_k|, x + 2|y_k|])}{4|y_k|} \leq \frac{4|y_k| - 2\epsilon_0|y_k|}{4|y_k|} = 1 - \frac{\epsilon_0}{2} < 1 \text{ for every k.}$$

A contradiction!

Exercise 10

Consider the function:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\to \sum_{n \in N^*} 2^{-n} \chi_{[q_n, \infty)}(x). \end{aligned}$$

Exercise 11

Proof. (1) When $a \leq b$, consider the partition $x_n := (\frac{\pi + 2n\pi}{2})^{-\frac{1}{b}}$ which is not of bounded variation.
When $a > b$, $f'(x) = ax^{a-1} sin(x^{-b}) - bx^{a-b-1} cos(x^{-b})$ is absolutely integrable on [0,1]. Then $f(x) = \int_0^x f'(t)dt$. □

Exercise 12

Fundamental analysis.

Exercise 13

Proof. We will prove it by contradiction. Pick $\epsilon < 1$ and then there exist $\delta > 0$ such that

$$\sum_{k=1}^{N} (b_k - a_k) < \delta \implies \sum_{k=1}^{N} |F(b_k) - f(a_k)| < \epsilon.$$

For such $\delta > 0$, since the Cantor set has measure zero, we can find a collection of intervals (x_k, y_k) (**we permit the interval of the form** $[0, a)$ **or** $(b, 1]$ **since it's open in the subspace toplogy**) that cover the Cantor points in $[0, 1]$ such that

$$\sum_k |y_k - x_k| < \delta$$

. Then we can find a finite subcover of the cover mentioned above since the cantor set is compact. We denote the index set by K. However, since the Cantor function only changes on the Cantor set,

$$\sum_{k \in K} |f(y_k) - f(x_k)| = 1$$

and absolute continuity is violated. □

Exercise 14

Proof. First extend the domain of F. Then observe that

$$\{D^+(F) < a\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\frac{F(x + \frac{1}{k} - F(x))}{\frac{1}{k}} < a\}$$

The second is very much alike. □

Exercise 15

Proof. Write $F = G_1 - G_2$ where G_1 and G_2 are increasing. Moreover an increasing function is a continuous increasing function plus a jump function. For example, $G_1 = F_1 + J_1$ where F_1 is continuous and increasing, and J_1 is a jump function; similarly, $G_2 = F_2 + J_2$.
Then $F = (F_1 - F_2) + (J_1 - J_2)$. But $J_1 - J_2$ **is a jump function**, and jump functions are continuous only if they're constant. Since F is continuous, $J_1 - J_2$ is constant; WLOG, $J_1 - J_2 = 0$ and then $F = F_1 - F_2$. □

Exercise 16

Proof. Since F is of bounded variation, $F(x) - F(a) = P_F(a,x) - N_F(a,x)$. Then define $G(x) := P_F(a,x) + F(a)$, $H(x) := N_F(a,x)$ and $g_n(x) := \frac{G(x+\frac{1}{n})-G(x)}{\frac{1}{n}} \geq 0$. Therefore

$$F'(x) = G'(x) - H'(x) \text{ a.e. x}$$

and

$$\int_{[a,b]} g_n(x) = \frac{1}{n}(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx) = \frac{1}{n}(\int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx) \leq G(b) - G(a).$$

By applying Fatou's lemma:

$$\int_a^b G'(x) = \int_a^b \lim_{n\to\infty} g_n(x) \leq \liminf_{n\to\infty}(G(b) - G(a)) = G(b) - G(a).$$

Similarly:

$$\int_a^b H'(x)dx \leq H(b) - H(a).$$

Then the conclusion is an easy consequence of the ineqaulity $|F'(x)| \leq G'(x) + H'(x)$ a.e. x. □

Exercise 17

Proof. Let $vol(B_1)$ denote the volume of B_1. Therefore

$$\begin{aligned}
|(f * K_\epsilon)(x)| &= \int_{\mathbb{R}^d} |f(x-y)K_\epsilon(y)|dy = (\int_{|y|\leq\epsilon} + \sum_{k=1}^\infty \int_{2^{k-1}\epsilon<|y|\leq 2^k\epsilon})|f(x-y)K_\epsilon(y)|dy \\
&\leq \frac{A}{\epsilon^d}\int_{|y|\leq\epsilon}|f(x-y)|dy + \sum_{k=1}^\infty \frac{A\epsilon}{(2^{k-1}\epsilon)^{d+1}}\int_{2^{k-1}\epsilon<|y|\leq 2^k\epsilon}|f(x-y)|dy \\
&\leq \frac{A}{\epsilon^d}\cdot vol(B_1)\cdot\epsilon^d f^*(x) + \sum_{k=1}^\infty \frac{A\epsilon}{(2^{k-1}\epsilon)^{d+1}}\cdot vol(B_1)\cdot(2^k\epsilon)^d\cdot f^*(x) \\
&= vol(B_1)(A + \sum_{k=1}^\infty 2^{kd-(k-1)(d+1)})f^*(x) = vol(B_1)(A + \sum_{k=1}^\infty 2^{d-k+1})f^*(x) \\
&< vol(B_1)(A + 2^{d+3})f^*(x).
\end{aligned}$$

We shall let C denote $vol(B_1)(A + 2^{d+3})$. □

Exercise 18

Proof. Think of the Cantor-Lebesgue function as the following process:
1) Given x, let y be the greatest member of the Cantor set such that $y \leq x$. (We know such a y exists because the Cantor set is closed.)
2) Write the ternary expansion of y.
3) Change all the 2's to 1's and re-interpret as a binary expansion. The value obtained is F(x).

It's pretty clear that both definitions of the Cantor-Lebesgue function given in the text do exactly this. □

Exercise 19

Proof. a) Assume $\epsilon > 0$. Since f is absolutely continuous, for such ϵ, there exist $\delta > 0$ such that

$$\sum_{k=1}^{N}(b_k - a_k) < \delta \implies \sum_{k=1}^{N}|F(b_k) - f(a_k)| < \epsilon.$$

Let E denote a set of measure 0. Then

$$m(E) = \inf_{\substack{E\subset O \\ O(open)}} m(O) \implies \exists \text{ open set } O \text{ such that } E \subset O \text{ and } m(O) < \delta.$$

It's clear that O can be written as $\bigcup_k (a_k, b_k)$ where each iterval is disjoint and $\sum_k |b_k - a_k| < \delta$.
Moreover here is a **fact:** let $[\hat{a}_k, \hat{b}_k] := f([a_k, b_k])$. Then $\exists\ m_k, n_k \subset [a_k, b_k]$ such that $f(m_k) = a_k$ and $f(n_k) = b_k$.
Since

$$f(E) \subset f(O) \subset f(\bigcup_k [a_k, b_k]) \subset \bigcup_k f([a_k, b_k]) = \bigcup_k [\hat{a}_k, \hat{b}_k] \subset \bigcup_k [f(m_k), f(n_k)],$$

$$m(f(E)) \leq \sum_k |f(m_k) - f(n_k) \leq \sum_k |f(b_k) - f(a_k)| < \epsilon.$$

(b) Since f **sends F_σ-set to F_σ-set and set of zero measure to set of zero measure**, and by applying the fact

$$f(A \cup B) = f(A) \cup f(B),$$

the conclusion is quite clear. □

Exercise 20

Proof. (a) Let $F(x) = \int_a^x \chi_K(x)dx$ where K denote the complement of a cantor-like set C of positive measure.

First we prove it's **strictly increasing**: Suppose $a \leq x < y \leq b$. By the construction of Cantor-like set, it's clear that $\exists (x', y') \subset K$ such that $(x', y') \subset (x, y)$. Therefore

$$\int_x^y \chi_K(x)dx \geq \int_{x'}^{y'} \chi_K(x)dx = y' - x' > 0$$

Second we prove $F'(x) = 0$ for a.e. $x \in C$. Suppose $x_0 \in C$ is differentiable and $\lim_{\substack{x_0 \in B(open) \\ m(B)\to 0}} \frac{m(B\cap K)}{m(B)} = 0$, then

$$\frac{1}{h}\int_{x_0}^{x_0+h} \chi_K(x)dx = \frac{1}{2h}\int_{x_0-h}^{x_0+h} \chi_K(x) = 0$$

Since such x_0 is a.e. in C, we prove the proposition.

(b) Consider $K = \bigcup_i (a_i, b_i)$, then $F(K) = \bigcup_i (F(a_i), F(b_i))$. Then

Fact 1: $m(F(K)) + m(F(C)) = m(F([a,b])) = m([A,B]) = B - A$

Fact 2: $B - A = F(b) - F(a) = \int_a^b \chi_K(x) = \sum_i \int_{a_i}^{b_i} \chi_K(x) = \sum_i (F(b_i) - F(a_i)) = m(F(K))$.

Combining Fact 1 and 2 we conclude $m(F(C)) = 0$, the following is trivial.

(c) **We first give some lemmas:**

Lemma 1: Let B be a Borel measurable set and f be a continuous function, then $f^{-1}(B)$ is a Borel set.

Lemma 2: Any function that has a derivative at every point of a set satisfies the Lusin (N) condition.

Lemma 1 is clear. To prove lemma 2, just think about why we want a bounded derivative and then consider splitting the set into pieces on which you do have a bounded derivative and then adding them up. Just observe the fact

Fact 1: Suppose $f'(x)$ exists at each point $x \in E$ and $|f'(x)| \leq M$. Then $m(f(E)) \leq Mm(E)$. Hence $f(E)$ is of measure zero if E is measure zero.

What's more, if E has measure zero and $f'(x)$ is finite at every point of E ($|f'(x)|$ not necessarily bounded) then simply write $E_n = \{x \in E : |f'(x)| \leq n\}$ and use the fact that

$$m(f(E)) \leq \sum_{n=1}^{\infty} m(f(E_n)) \leq \sum_{n=1}^{\infty} nm(E_n) = 0.$$

Suppose $E = H \cup Z$ where H is a F_σ-set and Z is of zero measure.
Then suppose Z is a zero-measure set, we claim that $F^{-1}(Z \bigcap \{F' > 0\})$ is of zero measure by lemma 2.
Then the conclusion is an easy consequence of the following fomula:

$$F^{-1}(H \cup Z) \bigcap \{F' > 0\} = \Big(F^{-1}(H) \cap \{F' > 0\}\Big) \bigcup \Big(F^{-1}(Z) \cap \{F' > 0\}\Big)$$

□

Exercise 21

Proof. (a) Observe that when $a > 0$ ($a \leq 0$ can be treated similarly):

$$\{f(F(x))F'(x) < a\} = \{F' = 0\} \bigcup \bigcup_{r_n \in \mathbb{Q}} \{(f(F(x)) < r_n) \wedge (\frac{a}{F'(x)} > r_n)\}.$$

(b) **First we show that the fomula below holds for all F_σ-sets:**

$$m(G) = \int_{F^{-1}(G)} F'(x)dx.$$

Suppose H be a H_σ-set. Then there is a decreasing sequence $\{C_n\}$ of closed sets such that $\bigcup_n C_n = G$ which implies that $m(H) = \lim_n m(C_n)$.

Now, $m(C_n) = \int_{F^{-1}(C_n)} F'(x)\ dx = \int_{[a,b]} \chi_{F^{-1}(C_n)}(x) \cdot F'(x)dx$,
so that, noting that $\chi_{F^{-1}(C_n)} \to \chi_{F^{-1}(H)}$ as $n \to \infty$, and that $|F'(x)|$ is bounded a.e. and integrable on $[a,b]$, an application of the *DCT* gives

$$\begin{aligned} m(H) = \lim_{n\to\infty} m(C_n) &= \lim \int_{[a,b]} \chi_{F^{-1}(C_n)}(x) \cdot F'(x)dx = \int_{[a,b]} \lim(\chi_{F^{-1}(C_n)}(x) \cdot F'(x))dx \\ &= \int_{[a,b]} \chi_{F^{-1}(H)}(x) \cdot F'(x)dx = \int_{F^{-1}(H)} F'(x)dx. \end{aligned}$$

Then for all measurable sets denoted by $E = H \cup Z$ where H is a F_σ-set and Z is a zero-measure set:

$$\begin{aligned} \int_A^B \chi_E(y)dy = \int_a^b \chi_{F^{-1}(E)}(x)F'(x)dx &= (\int_{\{F'>0\}} + \int_{\{F'=0\}})\chi_{F^{-1}(H\bigcup Z)}(x)F'(x)dx \\ &= \int_a^b \chi_{F^{-1}(E)}(x)F'(x)dx = \int_a^b \chi_E(F(x))F'(x)dx \end{aligned}$$

Then just extending the suited functions by **Thm 2.4.** □

Exercise 24

Proof. (a) Let $F_J(x)$ be the jump function associate with F. By lemma 3.13, $G(x) = F(x) - F_J(x)$ is increasing and continuous. Therefore, $G(x)$ is of BV, which implies that

$$\int_a^b G'(x)dx \le G(b) - G(a).$$

Then $G'(x)$ is an integrable function. Let $F_A(x) = \int_a^x G'(y)dy$. Therefore $F_A(x)$ is absolutely continuous and $F_A'(x) = G'(x)$ a.e.. Finally let $F_C(x) := G(x) - F_A(x)$ and then F_C is continuous and

$$F_C' = G' - F_A' = 0 \text{ a.e.}$$

Therefore

$$F(x) = G(x) + F_J(x) = (F_A + F_C + F_J)(x).$$

(b) Assume $F_A + F_C + F_J = \tilde{F}_A + \tilde{F}_C + \tilde{F}_J$.
Then

$$F_A' = \tilde{F}_A' \text{ a.e.} \implies F_A - \tilde{F}_A \text{ is a constant denoted by } C_1.$$

Then we take limits to cancel the effects of jump functions by paying attention to its property. □

Exercise 32

Proof. Necessity is trivial. Sufficiency follows by writing:

$$f(x) - f(y) = \int_y^x f'(t)dt.$$

□