Real Analysis: Chapter 2

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Some Words

I finished part of problems of chapter 1 and all exercises of chapter 2. Here follows them.

Problems of Chapter 1

Problem 1

Proof. Consider Borel-Cantelli lemma: Since $\{(m,n) : m,n \in \mathbb{Z},\ n \neq 0 \text{ and m,n are relatively prime}\}$ is countable, we enumerate it with $\{r_k\}_{k=1}^{\infty}$. Then we consider the events:

$$E_{r_k} = \{x : r_k = (m,n) \quad and \quad |x - \frac{m}{n}| \leq \frac{1}{n^3}\}.$$

Then It's clear that

$$\begin{aligned}
&\sum_{k=1}^{\infty} m\Big(E_{r_k} \bigcap [n, n+1)\Big) \leq \sum_{k=1}^{\infty} \frac{2}{k^3} \sum_{h=1}^{\infty} \frac{2}{h^2} < \infty \\
\Longrightarrow\ & m\bigg(\limsup_{k\to\infty} \Big(E_{r_k} \bigcap [n, n+1)\Big)\bigg) = 0 \\
\Longrightarrow\ & m\Big(\limsup_{k\to\infty} E_{r_k}\Big) = \sum_{n=-\infty}^{\infty} m\bigg(\limsup_{k\to\infty} \Big(E_{r_k} \bigcap [n, n+1)\Big)\bigg) = 0.
\end{aligned}$$

□

Problem 5

Proof. Let $\epsilon > 0$ be arbitrarily given and U_i, $i = 1, 2$ be open sets such that $E_i \subset U_i$ and

$$m_*(E_i) \leq m(U_i) < m_*(E_i) + \epsilon. \text{ for } i = 1, 2$$

Since $E \subset U_1 \cup U_2$, we have

$$mE \leq m(U_1 \cup U_2) = m(U_1) + m(U_2) - m(U_1 \cap U_2).$$

This gives $m(U_1 \bigcap U_2) \leq m(U_1) + m(U_2) - m(E) < 2\epsilon$ by the assumption that $mE = m_*(E_1) + m_*(E_2)$. Now, observe that

$$U_i \setminus E_i \subset \Big(U_1 \bigcap U_2\Big) \bigcup \bigg(\Big(U_1 \bigcup U_2\Big) \setminus E\bigg). \text{ for } i = 1, 2.$$

This implies for i=1,2:

$$\begin{aligned}
m_*(U_i \setminus E_i) &\leq m_*\bigg((U_1 \bigcap U_2) \bigcup \Big((U_1 \bigcup U_2) \setminus E\Big)\bigg) \\
&\leq m\Big(U_1 \bigcap U_2\Big) + m\Big((U_1 \bigcup U_2) \setminus E\Big) \\
&\leq 2\epsilon + m(U_1 \cup U_2) - m(E) \\
&\leq 2\epsilon + m(U_1) + m(U_2) - m(E) < 4\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it says that E_i are measurable for $i = 1, 2$. □

Problem 7

Proof. Problem: Show that if $f(x)$ is linear, then $m(\Gamma+\Gamma)=0$, and that if $f(x)$ is not linear, then $\Gamma+\Gamma$ contains an open set.
The set $\Gamma+\Gamma$ is $\{(x+z, f(x)+f(z) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. By just the form $y = mx+b$, it's easy to show that if $f(x)$ is linear then $C+C$ is on a line.
To prove the second part, consider the Jacobian determinant of

$$(x,z) \to (x+z, f(x)+f(z))$$

is $f'(z)-f'(x)$. This is going to be nonzero if $x \neq z$ are both near a point y where $f''(y) \neq 0$. So **the inverse function theorem** implies that the image of this map contains an open disc centered at the image of some such (x,z). □

Exercises of Chapter 2

exercise 1

Proof. Consider the collection:

$$\mathbb{F} = \left\{ \bigcap_{k=1}^{n} \widetilde{F_k} : \ \widetilde{F_k} \text{ denotes } F_k \text{ or } (F_k)^c. \right\} - \left\{ \bigcap_{k=1}^{n} (F_k)^c \right\},$$

which clearly holds the conditions in the problem. □

exercise 2

Suppose $h_\delta(x) := h(\delta x)$.

Proof. The proof is a simple consequence of the approximation of integrable functions by continuous functions of compact support as given in theorem 2.4.
For any $\epsilon > 0$, $\exists$ continuous function g with compact support such that $\|f-g\| < \epsilon$.
Since g is continuous and **has compact support** we have that clearly:

$$\|g_\delta - g\| = \int_{\mathbb{R}^d} |g(\delta x) - g(x)| dx \to 0 \quad \text{as } \delta \to 1.$$

Finally we have:

$$\begin{aligned} \|f_\delta - f\| &\leq \|g_\delta - g\| + \|f_\delta - g_\delta\| + \|f-g\| \\ &\leq \|g_\delta - g\| + \frac{\|f-g\|}{\delta^d} + \|f-g\| \to 0 \quad \text{as } \delta \to 1. \end{aligned}$$

So if $|\delta| < \Delta$, where Δ is a potive number sufficiently small, then

$$\|g_\delta - g\| < \epsilon \quad \textit{and} \quad \frac{\|f-g\|}{\delta^d} < \epsilon,$$

which follows that $\|f_\delta - f\| < 3\epsilon$ whenever $|\delta| < \Delta$. □

exercise 3

Proof. First we point out an equation: for any interval $E \in \mathbb{R}$

$$\begin{aligned}\int_{E+2\pi} f(x)dx &= \int_{\mathbb{R}} \chi_{E+2\pi}(x)f(x)dx = \int_{\mathbb{R}} \chi_E(x-2\pi)f(x)dx = \int_{\mathbb{R}} \chi_{-E}(2\pi - x)f(x)dx \\ &= \int_{\mathbb{R}} \chi_{-E}(x)f(2\pi - x)dx = \int_{\mathbb{R}} \chi_E(-x)f(-x)dx = \int_{\mathbb{R}} \chi_E(x)f(x)dx \\ &= \int_E f(x)dx.\end{aligned}$$

WLOG, we suppose $I := (a,b)$. (**Containing endpoints or not doesn't matter.**) Also it's clear that I is contained in two consective intervals of the form $(k\pi, k\pi + 2\pi)$ and $(k\pi + 2\pi, k\pi + 4\pi)$. Then we have

$$\begin{aligned}\int_I f(x)dx &= (\int_{(a,k\pi+2\pi)} + \int_{(k\pi+2\pi,b)})f(x)dx \\ &= \int_{(k\pi,k\pi+2\pi)} f(x)dx = \int_{(0,2\pi)} f(x)dx \\ &= (\int_{(0,\pi)} + \int_{(\pi,2\pi)})f(x)dx = (\int_{(0,\pi)} + \int_{(-\pi,\pi)})f(x)dx \\ &= \int_{(-\pi,\pi)} f(x)dx.\end{aligned}$$

□

exercise 4

Proof. WLOG we assume that $f(t) \geq 0$. Now suppose

$$\widetilde{f}(t) = \begin{cases} f(t) & \text{if } 0 < t \leq b \\ 0 & \text{if else} \end{cases} \quad and \quad \widetilde{g}(t) = \begin{cases} \frac{1}{t} & \text{if } t > 0 \\ 0 & \text{if else} \end{cases}. \tag{1}$$

It's clear that $\widetilde{f}(t)$ and $\widetilde{g}(t)$ are measurable functions on $\mathbb{R}$. Then we let

$$h(x,t) = \widetilde{f}(t)\widetilde{g}(t)\chi_{\{0<x\leq t\leq b\}} \tag{2}$$

Then $h \geq 0$ and h is clearly measurable since they are multiples of three measurable functions.
By Fubini's theorem, the function $\int_{-\infty}^{\infty} h(x,t)dt$ is measurable on $\mathbb{R}$.
Moreover we have:

$$\int_{-\infty}^{\infty} h(x,t)dt = \int_x^b h(x,t)dt = \int_x^b \frac{f(t)}{t}dt = g(x), \text{where } 0 < x \leq b.$$

Therefore we have:

$$\begin{aligned}\int_0^b g(x)dx &= \int_0^b \left(\int_{-\infty}^{\infty} h(x,t)dt\right)dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(x,t)dt\right)dx = \int_{\mathbb{R}^2} h(x,t) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(x,t)dx\right)dt = \int_0^b \left(\int_0^t h(x,t)dx\right)dt = \int_0^b f(t)dt,\end{aligned}$$

It follows that g(x) is measurable on $[0,b]$. □

exercise 5

We shall omit it since it's already taught as an example in the class.

exercise 6

(a) **Solution**: Suppose a function:

$$f(t) = \begin{cases} n & \text{if } t \in [n, n+\frac{1}{n^3}), n = 1,2,3,\ldots \\ 0 & \text{if else} \end{cases} \tag{3}$$

It's clear that $f(t)$ is measurable but $\limsup_{x\to\infty} f(x) = \infty$.
(b) **Solution**: We prove by contradiction. If $\lim_{|x|\to\infty} f(x) \neq 0$, then by definition $\exists \epsilon > 0$ such that

$$\exists \{x_n\} \text{ s.t. } |x_n| \to \infty \text{ and } |f(x_n)| > \epsilon.$$

WLOG, we suppose $|x_{n+1}| > |x_n| + 1$ for every n in the sequence.
Since f is uniformly continuous, for such ϵ,

$$\exists 0 < \delta < \frac{1}{2}, \forall |x-y| < \delta \left(|f(x) - f(y)| < \frac{\epsilon}{2} \right).$$

Then it's clear that

$$(x_n - \frac{1}{2}, x_n + \frac{1}{2}) \bigcap \{x_m - \frac{1}{2}, x_m + \frac{1}{2}) = \emptyset \text{ for every } m, n \in \mathbb{N}$$

and

$$(x_n - \frac{1}{2}, x_n + \frac{1}{2}) \subset \{x : f(x) > \frac{\epsilon}{2}\} \text{ for every } n \in \mathbb{N}.$$

Therefore by Tchebychev inequality:

$$\int |f| \geq \frac{\epsilon}{2} \times m\left(\{x : f(x) > \frac{\epsilon}{2}\} \right) = \infty$$

A contradiction from the integrability of the function.

exercise 7

Proof. We let $F(x,y) = y - f(x)$ which is a measurable function by corollary 3.7. Then $\{F = 0\} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\} = \gamma$ is measurable, which proves the first assertion.
Suppose $g := \chi_\Gamma$ on $\mathbb{R}^{d+1}$. By Tonelli's theorem, we have

$$m(\gamma) = \int_\gamma \chi_\Gamma = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} g_x dy \right) dx = \int_{\mathbb{R}^d} 0 dx = 0$$

$\square$

exercise 8

Proof. Let $\epsilon > 0$. By the **absolute continuity of the integral**, $\exists \delta > 0$ such that

$$m(E) < \delta \implies \int_E |f| < \epsilon.$$

Then for such $\epsilon > 0$, we clearly have

$$\forall |y - \tilde{y}| < \delta \left(|F(\tilde{y}) - F(y)| = |\int_{\tilde{y}}^{y} f dx| < \epsilon \right),$$

since $m([y, \tilde{y}]) < \delta$. (WLOG, we suppose $y \leq \tilde{y}$) $\square$

exercise 9

Proof. Since $\alpha \chi_{E_\alpha} \leq f$, we have

$$\int \alpha \chi_{E_\alpha} \leq \int f \implies m(E_\alpha) \leq \frac{1}{\alpha} \int f.$$

$\square$

exercise 10

Proof. suppose

$$g(x) = \sum_{k=-\infty}^{\infty} 2^k \chi_{F_k} \quad and \quad h(x) = \sum_{k=-\infty}^{\infty} 2^k \chi_{E_{2^k}}.$$

Then it's clear that $g \leq f \leq h$, which follows that

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \int g \leq \int f \leq \int h = \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}). \tag{4}$$

Also we have the following:

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \implies \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{k} 2^h m(F_k) < \infty \implies \sum_{h=-\infty}^{\infty} \sum_{k=h}^{\infty} 2^h m(F_k) < \infty$$
$$\implies \sum_{h=-\infty}^{\infty} 2^h m(E_{2^h}) < \infty.$$

From (4) and the induction above:

$$\text{f is integrable iff} \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \text{ iff} \sum_{h=-\infty}^{\infty} 2^h m(E_{2^h}) < \infty.$$

Then we apply the proposition we proved. First for $f(x)$:

$$E_{2^k} = O\left(0, \frac{1}{2^{\frac{k}{a}}}\right) \quad and \quad F_k = O\left(0, \frac{1}{2^{\frac{k}{a}}}\right) - O\left(0, \frac{1}{2^{\frac{k+1}{a}}}\right)$$

Thus

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = C \sum_{k=-\infty}^{\infty} 2^k (\frac{1}{2^{\frac{k}{a}}})^d < \infty \Leftrightarrow 1 - \frac{d}{a} < 0 \Leftrightarrow a < d.$$

Simlarly we can prove the case in $g(x)$. $\square$

exercise 11

Proof. We prove by contradiction. If the statement ($f(x) \geq 0$ a.e. x.) doesn't hold, then We have the contradiction:

$$\begin{aligned} m(\{f<0\})>0 &\implies \lim_{n\to\infty} m(\{f<-\frac{1}{n}\}) = m(\{f<0\})>0 \\ &\implies \exists N \text{ s.t. } m(\{f<-\frac{1}{N}\})>0. \\ &\implies \int_{\{f<-\frac{1}{N}\}} f dx \leq -\frac{m(\{f<-\frac{1}{N}\})}{N} < 0. \end{aligned}$$

Then we have

$$\begin{aligned} \int_E f dx = 0 \text{ for a.e. x} &\implies f(x) \geq 0 \text{ for a.e. x} \quad \textit{and} \quad f(x) \leq 0 \text{ for a.e. x.} \\ &\implies f(x) = 0 \text{ for a.e. x.} \end{aligned}$$

□

exercise 12

From exercise 14 of chapter 2 in $\mathbb{R}^d$ we know that $m(B(0,r)) = Cr^d$ where C is a constant dependent on the dimension of $\mathbb{R}^d$.

Proof. We shall construct a sequence $\{I_n\}$ such that for any $x \in \mathbb{R}^d$, there are infitely many I_n containing x. Observe that $\exists\{\mathrm{N}_k : k = 1, 2, \ldots\}$ such that:

$$N_1 = 1 \quad \textit{and} \quad \sum_{k=N_i}^{\mathrm{N}_{i+1}-1} \frac{1}{k} > i.$$

Suppose $T_j = \sum_{i=1}^{j} N_i$. Then we let

$$I_n = \begin{cases} B\left(0, (\frac{1}{N_j})^{\frac{1}{d}}\right) & \text{if } n = T_j \ (j = 1, 2, \ldots) \\ B\left(0, (\sum_{k=N_j}^{N_j+(n-T_j)} \frac{1}{k})^{\frac{1}{d}}\right) - B\left(0, (\sum_{k=N_j}^{N_j+(n-T_j-1)} \frac{1}{k})^{\frac{1}{d}}\right) & \text{if } T_j < n < T_{j+1} \end{cases}. \tag{5}$$

Then it's clear that for I_n: **for any $x \in \mathbb{R}^d$, there are infinitely many I_n containing x**.
Then we let $f = 0$ and $f_n = \chi_{I_n}$. Since for any $x \in \mathbb{R}^d$, there are infinitely many I_n containing x,

$$f_n(x) \to f(x) \text{ for no x.}$$

Moreover since $|I_n| \to 0$,

$$\|f - f_n\|_{L^1} \to 0.$$

□

exercise 13

Proof. Let $A = \{0\} \times [0,1]$ and $B = \mathcal{N} \times \{0\}$. It's clear that

$$m^*(A) = m^*(B) = 0 \implies A, B \text{ measurable.}$$

While $A + B = \mathcal{N} \times [0,1]$ is not measurable, since otherwise

$$\mathcal{N} \times [0,1] \text{ measurable} \quad \textit{and} \quad m^*([0,1]) > 0 \implies \mathcal{N} \subset \mathbb{R} \text{ measurable.}$$

A contradiction! □

exercise 14

Proof. (a) Define a measurable function (its measurable is from its continuity.)

$$f(x) = \begin{cases} (1-x^2)^{\frac{1}{2}} & \text{if } x \in [-1,1]. \\ 0 & \text{if else.} \end{cases} \tag{6}$$

Let $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}$. Since f is measurable on $\mathbb{R}$, A is measurable by corollary 3.8. Moreover,

$$\begin{aligned} v_2 &= m\left(A \bigcup B\right) = m(A) + m(-A) = 2m(A) = 2\int_{\mathbb{R}^2} \chi_A \\ &= 2\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_A dy\right) dx = 2\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} dx. \end{aligned}$$

(b) Similarly, Let $A = \{(x,y) : \mathbb{R}^{d-1} \times \mathbb{R} : 0 \leq y \leq f(x)\}$, with

$$f(x) = \begin{cases} (1-|x|^2)^{\frac{1}{2}} & \text{if } |x| \in [0,1]. \\ 0 & \text{if else.} \end{cases} \tag{7}$$

Similarly we have the following:

$$\begin{aligned} v_d &= m\left(A \bigcup B\right) = m(A) + m(-A) = 2m(A) = 2\int_{\mathbb{R}^d} \chi_A = 2\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} \chi_A(x,y) dx\right) dy \\ &= 2\int_{[0,1]} (1-y^2)^{\frac{d-1}{2}} \left(\int_{\mathbb{R}^{d-1}} \chi_A((1-y^2)^{\frac{1}{2}} x, y) dx\right) dy \\ &= 2\int_{[0,1]} (1-y^2)^{\frac{d-1}{2}} m\left(\{x \in \mathbb{R}^{d-1} : f((1-y^2)^{\frac{1}{2}} x) \geq y\}\right) dy \\ &= 2\int_{[0,1]} (1-y^2)^{\frac{d-1}{2}} m\left(\{x \in \mathbb{R}^{d-1} : |x| \leq 1\}\right) dy \\ &= 2v_{d-1} \int_0^1 (1-x^2)^{\frac{d-1}{2}} dx. \end{aligned}$$

(c) We shall omit it since it's usual procedure of fundamental analysis.

□

exercise 15

Proof. Since $f(x)$ is riemann integrable on $[0,1]$, f is lebsgue-integrable on [0,1] by theorem 1.5. It follows that:

$$\int_{\mathbb{R}}^{L} f < \infty \implies f \text{ is integrable on } \mathbb{R}.$$

Therefore for every $n \in \mathbb{N}^*$, $f(x - r_n)$ is integrable. And so is $\frac{f(x-r_n)}{2^n}$.
Since

$$(1) : \sum_{n=1}^{N} 2^{-n} f(x - r_n) \to F(x) \text{ a.e. x} \quad \text{and} \quad (2) : F_N(x) := \sum_{n=1}^{N} 2^{-n} f(x - r_n) \text{ s.t. } 0 \le F_N \le F,$$

by Monotone Convergence Theorem, we have:

$$\int_{\mathbb{R}} F = \lim_{N\to\infty} \int_{\mathbb{R}} F_N = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{\mathbb{R}} 2^{-n} f(x - r_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} < \infty \implies F \text{ is measurable.}$$

This implies that F is finite-valued for almost all $x \in \mathbb{R}$.
Now let $\widetilde{F}$ be any function that agrees with F almost everywhere, and I be any interval on the real line. Let r_N be some rational number contained in I. Then for any $M > 0$,

$$f(x - r_N) > M, \; x \in (r_N - \frac{1}{2^N M^2}, r_N + \frac{1}{2^N M^2})$$

which intersects I in an interval I_M of positive measure. Since $\widetilde{F}$ agrees with F a.e., it must also be greater than M a.e. in this interval $I_M \subset I$. Hence $\widetilde{F}$ exceeds any finite value M on I. □

exercise 16

Proof. WLOG, suppose $f \ge 0$. We only prove the case when $d = 2$, then it's an easy consequence of induction.

$$\begin{aligned}\int_{\mathbb{R}^2} f^\delta &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\delta_1 x_1, \delta_2 x_2) dx_1 \right) dx_2 = \frac{1}{|\delta_1|} \left(\int_{\mathbb{R}} f(x_1, \delta_2 x_2) dx_1 \right) dx_2 \\ &= \frac{1}{|\delta_1 \delta_2|} \left(\int_{\mathbb{R}} f(x_1, x_2) dx_1 \right) dx_2 = \frac{1}{|\delta_1 \delta_2|} \int_{\mathbb{R}^2} f(x_1, x_2) < \infty.\end{aligned}$$

□

exercise 17

Proof. (a) Since each slice f^y is a simple function, f^y is integrable. And so is f_x. For fixed x, we let $x \in [n, n+1)$ without loss of generality. Then

$$\int f_x(y) dy = \left(\int_{[n,n+1)} + \int_{[n+1,n+2)} \right) f_x(y) dy = a_n - a_n = 0 \implies \int \left(\int f(x,y) dy \right) dx = 0.$$

(b) First if $0 \le y < 1$, we have

$$\int f^y(x) dx = \int_{[0,1)} f^y(x) dx = a_0 \ge 0.$$

Problem 17 continued on next page...

Second if $n \le y < n+1$, we have

$$\int f^y(x)dx = \left(\int_{[n,n+1)} + \int_{[n-1,n)}\right) f^y(x)dx = a_n - a_{n-1} \ge 0.$$

Therefore we have

$$\int f^y(x)dx \ge 0 \quad \text{and} \quad \int \left(\int f^y(x)dx\right)dy = \sum_{k=0}^{\infty} b_k = s < \infty$$

(c) We have the following equation

$$\int_{\mathbb{R}^2} |f| = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f|dx\right)dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f dx\right)dy = s.$$

□

exercise 18

Proof. let

$$\widetilde{f}(t) = \begin{cases} f(t) & \text{if } 0 \le t \le 1 \\ 0 & \text{if else} \end{cases}. \tag{8}$$

Then let $g(x,y) := |\widetilde{f}(x) - \widetilde{f}(y)|$ which is measurable on $\mathbb{R}^2$. Since $|f(x) - f(y)|$ is integrable on $[0,1] \times [0,1]$,

$$\int_{\mathbb{R}^2} g(x,y) = \int_{[0,1]\times[0,1]} |f(x) - f(y)| < \infty$$

which means that g(x,y) is integrable on $\mathbb{R}^2$. By Fubini's lemma,

$$\begin{aligned} \int_{\mathbb{R}^2} g(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x,y)dx\right)dy &= \int_{[0,1]} \left(\int_{[0,1]} |f(x) - f(y)|dx\right)dy < \infty \\ \implies \exists\, y_0 \in [0,1] : &\int_{[0,1]} |f(x) - f(y_0)|dx < \infty \quad \textit{and} \quad |f(y_0)| < \infty. \\ \implies \int_{[0,1]} |f(x)|dx \le &\int_{[0,1]} |f(x) - f(y_0)|dx + |f(y_0)| < \infty. \end{aligned}$$

Therefore f is integrable on $[0,1]$. □

exercise 19

Proof. Since $\chi_{E_\alpha}(x,\alpha) \ge 0$ is measurable on $\mathbb{R}^d \times \mathbb{R}$, we have

$$\begin{aligned} \int_0^\infty m(E_\alpha)d\alpha = \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{E_\alpha} dx\right)d\alpha &= \int_{\mathbb{R}^d} \left(\int_0^\infty \chi_{E_\alpha} d\alpha\right)dx \\ &= \int_{\mathbb{R}^d} |f(x)|dx. \end{aligned}$$

□

exercise 20

Proof. To prove E^y is borel, consider a function $f(x) = (x, y)$. Since $E^y = f^{-1}(E \bigcap \{y\})$ and it's clear that f is a **borel function**, E^y is borel. $\square$

exercise 21

Proof. (a) Since $f(x-y)$ is measurable on $\mathbb{R}^{2d}$ by proposition 3.9 and $g(y)$ is measurable on $\mathbb{R}^{2d}$ by corollary 3.7, $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.
(b) The integrability of $f(x-y)g(y)$ is direct from below:

$$\begin{aligned}\int_{\mathbb{R}^{2d}} |f(x-y)g(y)| &= \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(x-y)|dx \right) dy = \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(x)|dx \right) dy \\ &= \int_{\mathbb{R}^d} |g(y)|dy \int_{\mathbb{R}^d} |f(x)|dx < \infty\end{aligned}$$

(c) We only need to prove

$$\int_{\mathbb{R}^d} |f(x-y)g(y)|dy < \infty \text{ for a.e. x},$$

which is clear by Fubini's lemma from

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)| < \infty.$$

(d) Since f and g are integrable, then $f * g$ is well-defined for a.e. x.
And we have

$$\begin{aligned}\int_{\mathbb{R}^d} |f| < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} |g| < \infty &\Rightarrow \int_{\mathbb{R}^{2d}} |f(x-y)g(y)| = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)g(y)|dy \right) dx < \infty \\ &\Rightarrow \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y)dy \right| dx \leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)g(y)|dy \right) dx < \infty\end{aligned}$$

,

therefore $f * g$ is integrable.
And we have

$$\begin{aligned}\|(f*g)\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |(f*g)(x)|dx \leq \int_{\mathbb{R}^{2d}} |f(x-y)g(y)| \\ &= \int_{\mathbb{R}^d} |g(y)|dy \int_{\mathbb{R}^d} |f(x)|dx = \|(f)\|_{L^1(\mathbb{R}^d)} \|(g)\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

(e) First it's **bounded**, since

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(x)| = \|f\|_{L^1(\mathbb{R}^d)} < \infty.$$

Second we prove it's **continuous**. Assume $\epsilon > 0$. Observaing:

$$|\hat{f}(\xi) - \hat{f}(\mu)| \leq \int_{\mathbb{R}^d} |f(x)||e^{-2\pi i x(\xi-\mu)} - 1| \tag{9}$$

We have two following facts

$$\text{for } \epsilon > 0, \exists R > 0 \left(\int_{B^c_{\mathbb{R}}} |f| < \frac{\epsilon}{4} \right). \quad \text{and} \quad |e^{i\theta} - 1| \leq |cos\theta| + |sin\theta|. \ (\theta \in \mathbb{R})$$

Then let $\|\xi-\mu\| < \frac{\epsilon}{8\pi R\|f\|_{L^1(\mathbb{R}^d)}}$. By applying the facts we observe:

$$\begin{aligned}|\hat{f}(\xi)-\hat{f}(\mu)| &\leq \int_{\mathbb{R}^d}|f(x)||e^{-2\pi ix(\xi-\mu)}-1| = \left(\int_{B_{\mathbb{R}}}+\int_{B^c_{\mathbb{R}}}\right)|f(x)||e^{-2\pi ix(\xi-\mu)}-1| \\ &\leq \int_{B_{\mathbb{R}}}|f(x)|\Big(|cos(2\pi x(\xi-\mu)|+|sin(2\pi ix(\xi-\mu)|\Big)+2\int_{B^c_{\mathbb{R}}}|f(x)| < C(\epsilon).\end{aligned}$$

Finally, the last equation follows from

$$\begin{aligned}\widehat{(f*g)}(\xi) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}f(x-y)g(y)dy\right)e^{-2\pi ix\xi}dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}f(x-y)e^{-2\pi i\xi(x-y)}g(y)e^{-2\pi i\xi y}dy\right)dx \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}f(x-y)e^{-2\pi i\xi(x-y)}dx\right)g(y)e^{-2\pi i\xi y}dy = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}f(x)e^{-2\pi i\xi x}dx\right)g(y)e^{-2\pi i\xi y}dy \\ &= \widehat{f}(\xi)\widehat{g}(\xi).\end{aligned}$$

□

exercise 22

Proof. Let $\xi' = \frac{1}{2}\frac{\xi}{|\xi|^2}$ and we have:

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{2}\Big(\int_{\mathbb{R}^d}f(x)e^{-2\pi ix\xi}dx+\int_{\mathbb{R}^d}f(x-\xi')e^{-2\pi i(x-\xi')\xi}dx\Big) \\ &= \frac{1}{2}\Big(\int_{\mathbb{R}^d}f(x)e^{-2\pi ix\xi}dx-\int_{\mathbb{R}^d}f(x-\xi')e^{-2\pi ix\xi}dx\Big) \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\Big(f(x)-f(x-\xi')\Big)e^{-2\pi ix\xi}dx \to 0 \quad \text{as } |\xi'|\to\infty.\end{aligned}$$

□

exercise 23

Proof. We will prove it by contradiction let $f = e^{-x^2}$, we have:

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty}e^{-\xi^2}e^{-i\lambda\xi} = e^{-\frac{\lambda^2}{4}}\int_{\mathbb{R}^d}e^{-t^2}dt = C(d)e^{-\frac{\lambda^2}{4}},$$

with $C(d)$ being a constant related to d.
Moreover

$$f*I = f \implies \widehat{f}\circ\widehat{I} = \widehat{f} \implies \widehat{I} = 1 \text{ for a.e. x},$$

a contradiction to exercise 22 since $\widehat{I}\to 0$ as $|\xi|\to\infty$. □

exercise 24

Proof. (a) Assume f is integrable and $\exists M \geq 0$ such that $|g| \leq M$. Then we have:

$$\begin{aligned}|(f*g)(x)-(f*g)(z)| &= \left|\int_{\mathbb{R}^d}\Big(f(x-y)-f(z-y)\Big)g(y)dy\right| \leq M\int_{\mathbb{R}^d}\Big|f(x-y)-f(z-y)\Big|dy \\ &= M\int_{\mathbb{R}^d}\Big|f(-y)-f(z-x-y)\Big|dy = M\|(f(y)-f(y-z+x))\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Then by proposition 2.5:

$$\forall\epsilon>0, \exists\delta>0 \text{ s.t. } \Big(\|z-x\|<\delta \implies \|f(y)-f(y-z+x)\|<\epsilon\Big).$$

It immediately follows that $f*g$ is uniformly continuous.
(b) Since f and g are in $L^1(\mathbb{R}^d)$, $f*g$ is in $L^1(\mathbb{R}^d)$ by exercise 21. Since $f*g$ is uniformly continuous and integrable and exercise 6(b), we have

$$\lim_{|x|\to\infty}(f*g)(x)=0.$$

□

exercise 25

We shall omit it since hint tells everything.