Real Analysis: Stein Chapter 1

Due on April 13, 2021 at 24:00pm

Professor Lilu Zhao Week 6

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Problem 1

Addition Problem: If theorem 4.4 has no condition '$m(E) < \infty$', does the conclusion still hold?
Solution Not hold!!

Proof. Assume $\forall n \geq 1 : f_n : [0, \infty) \to \{0, 1\}$ with $f_n := \chi_{[n-1,n]}$. Then $f_n \to 0$ pointwise on $\mathbb{R}$. Suppose $\exists F \subseteq \mathbb{R} : f_n \xrightarrow{u} 0$ on F, in other words we have:

$$\forall \epsilon > 0, \exists N, \forall n \geq N, \forall x \in F : |f_n(x)| < \epsilon.$$

For example, suppose $\epsilon := 1, \exists N, \forall n \geq N, \forall x \in F : |f_n(x)| < 1$. Thus $x \notin [N, \infty)$. Thus $F \subseteq [0, N)$, and consequently $m(\mathbb{R} - F) \geq m([N, \infty)) = \infty$. A contradiction! $\square$

Problem 17

Proof. First we clearly have the following equation for every $n \in N^*$:

$$\bigcap_{k=1}^{\infty} \{x : |f_n(x)| > \frac{k}{n}\} = \{x : |f_n(x)| = \infty\},$$

which follows that for every $n \in N^*$ we have

$$m(\bigcap_{k=1}^{\infty} \{x : |f_n(x)| > \frac{k}{n}\}) = m(\{x : |f_n(x)| = \infty\}) = 0$$

with the second equation follows from the condition of the problem.
Then since the sequence $\{x : |f_n(x)| > \frac{k}{n}\}$ is monotone for parameter k and $m(\{x : |f_n(x)| > \frac{1}{n}\}) < \infty$, it follows that corollary 3.3 that

$$\lim_{k \to \infty} m(\{x : |f_n(x)| > \frac{k}{n}\}) = m(\bigcap_{k=1}^{\infty} \{x : |f_n(x)| > \frac{k}{n}\}) = 0.$$

Hence for every n we have the following equation

$$\lim_{k \to \infty} m(\{x : |\frac{f_n(x)}{k}| > \frac{1}{n}\}) = 0.$$

From the equation above we certainly for every n we have a $c_n \in N^*$ such that

$$m(\{x : |\frac{f_n(x)}{c_n}| > \frac{1}{n}\}) < \frac{1}{2^n}.$$

Then we define $E_n = \{x : |\frac{f_n(x)}{c_n}| > \frac{1}{n}\}$. Thus since

$$\sum_{n=1}^{\infty} m(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty,$$

it follows from **Borel-Cantelli lemma** that

$$m(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n) = 0.$$

By taking the complement operation we have

$$m(\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} (E_n)^c) = 1$$

Problem 17 continued on next page...

which means that the following holds

$$\frac{f_n(x)}{c_n} \to 0 \qquad a.e. \quad x.$$

□

Problem 18

We first prove the simple case in $\mathbb{R}$.

Proof. Assume E be some measurable set in $\mathbb{R}$. Let f: $E \to \bar{\mathbb{R}}$ be measurable. Let $B_n = [-n, n] \bigcap E$. Then by Lusin's Theorem, there exists a compact subset $E_n \subset B_n$ with

$$m(B_n - E_n) < \frac{1}{2^n}$$

and f continuous on E_n. Then we will extend $f|_{E_n}$ to a continuous function f_n on all of $\mathbb{R}$, where $f_n = f$ on E_n. (It's in fact is **Tietze's Extension Theorem**). If the construction holds, then for every n we have $f_n|_E$ is continous from the definition of continuity.
Since $R - E_n$ is open in $\mathbb{R}$, we have $R - E_n$ can be written uniquely as countable union of disjoint open intervals, say

$$\mathbb{R} - E_n = \bigcup_{x \in (R - E_n)} I_x.$$

Then we define

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in E_n \\ f(a) & \text{if } I_x = (-\infty, a) \\ f(b) & \text{if } I_x = (b, \infty) \\ f(a) + \frac{x-a}{b-a} * b & \text{if } I_x = (a, b) \end{cases}. \tag{1}$$

Then it's clear that $f_n(x)$ is continuous on $\mathbb{R}$. Finally we assert that

$$f_n(x) \xrightarrow{a.e.} f(x).$$

It's clear that

$$\Big(E - \{x \in E : f_n(x) \to f(x)\}\Big) \subset \limsup_{n\to\infty}(B_n - E_n),$$

which follows that,

$$m\Big(E - \{x \in E : f_n(x) \to f(x)\}\Big) \le m\Big(\limsup_{n\to\infty}(B_n - E_n)\Big) \overset{\textbf{Borel-Cantelli}}{=\!=\!=\!=\!=} 0.$$

In other words we have

$$f_n(x) \xrightarrow{a.e.} f(x), x \in E.$$

□

We now prove the **general** case in $\mathbb{R}^n$.

Proof. We give a lemma first.
lemma (Tietze's Extension Theorem): Continuous functions on a closed subset of a normal topological space can be extended to the entire space.

Since $\mathbb{R}^n$ is normal, we can apply the theorem to closed subsets in $\mathbb{R}^n$. Moreover we define $C_m \triangleq \{x = (x_1, x_2, \ldots, x_n) : |x_i| \le m$ for every i$\}$.

Assume E be some measurable set in $\mathbb{R}^n$. Let f: $E \to \bar{\mathbb{R}}^n$ be measurable. Let $B_n = D_n \bigcap E$. Then by **Lusin's Theorem**, there exists a compact subset $E_n \subset B_n$ with

$$m(B_n - E_n) < \frac{1}{2^n}$$

and f continuous on E_n. Then we will extend $f|_{E_n}$ to a continuous function f_n on $\mathbb{R}^n$ by **Tietze's Extension Theorem**. Finally we assert that

$$f_n(x) \xrightarrow{a.e.} f(x).$$

It's clear that

$$\Big(E - \{x \in E : f_n(x) \to f(x)\}\Big) \subset \limsup_{n\to\infty}(B_n - E_n),$$

which follows that,

$$m\Big(E - \{x \in E : f_n(x) \to f(x)\}\Big) \le m\Big(\limsup_{n\to\infty}(B_n - E_n)\Big) \xlongequal{\textbf{Borel-Cantelli}} 0.$$

In other words we have

$$f_n(x) \xrightarrow{a.e.} f(x), x \in E.$$

□

Problem 19

Solution Define $O(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\} \subset \mathbb{R}^d$
(a) we will prove it with definition.

Proof. WLOG, we assume A is open. Suppose $x \in A + B$. Then from the definition of $A + B$ we have

$$x = x_1 + x_2 \qquad s.t. \quad x_1 \in A \quad and \quad x_2 \in B.$$

Since $x_1 \in A$, we have an open ball $O(x_1, r) \subset A$ where $r > 0$ and $x_1 \in O(x_1, r)$. The only thing we need to prove is that $O(x_1, r) + x_2$ is open, which is clear from the following equation

$$O(x_1, r) + x_2 = O(x_1 + x_2, r).$$

□

(b) We will prove it with the property of measurable sets.

Proof. We only need to show that $A + B$ is F_σ, which immediately follows that $A + B$ is measurable. First we prove that if A and B both are compact we have $A + B$ is compact.
As A and B are bounded, there exists $M_1, M_2 > 0$ such that:

$$|x_1| \le M_1 \text{ and } |x_2| \le M_2, \quad \forall x_1 \in A, \forall x_2 \in B.$$

Therefore for any $x = x_1 + x_2 \in A + B$, we have

$$|x| \le |x_1| + |x_2| \le M_1 + M_2$$

and thus S is bounded.
Next we are to prove it's closed. For any sequence $\{x_n\}_{n\ge1}$ in $A+B$ which converges to $x_* \in \mathbb{R}^d$, there exists two sequences $\{x_{1,n_j}\}_{j\ge1}$ and $\{x_{2,n_j}\}_{j\ge1}$ such that $x_{n_j} = x_{1,n_j} + x_{2,n_j}$ for every j, which converge to some $x_{1,*} \in A$ and $x_{2,*} \in B$ respectively. (Since A and B are compact sets, for any sequence in A or B there exist

a convergent subsequence. For example, assume $x_n = x_{1,n} + x_{2,n}$ where $x_{1,n} \in A$ and $x_{2,n} \in B$. We can first take the convergent subsequence of $x_{1,n}$. Then for the subsequence in $x_{2,n}$ with the same subscript as the convergent subsequnce of $x_{1,n}$ described above, we can take a subsequence which converges to B likewise. By adjusting the subscript we can construct the sequence described above.)
Hence we just have

$$x_{n_j} \to x_{1,*} + x_{2,*}, \quad j \to \infty.$$

which follows that $x_* = x_{1,*} + x_{2,*}$ belongs to $A + B$.
Then we define $E_n^{d+1} = \{x \in \mathbb{R}^d : |x| \leq n\}$. Since

$$A + B = \bigcup_{n=1}^{\infty} \left((A \bigcap E_n^{d+1}) + (B \bigcap E_n^{d+1}) \right)$$

and $(A \bigcap E_n^{d+1}) + (B \bigcap E_n^{d+1})$ is closed from the text above (**more precisely it is compact**), we have $A + B$ is a F_σ set. □

(c) For example, in $\mathbb{R}^2$, let $A = [0,1] \times \{0\}$ and $B = \{0\} \times \left(\mathbb{Q} \bigcap [0,1] \right)$. Thus we have

$$A + B = \left\{ [0,1] \times r : r \in \mathbb{Q} \bigcap [0,1] \right\}.$$

Let $x \in \left((\mathbb{R} - \mathbb{Q}) \bigcap [0,1] \right) \times [0,1]$. It's clear x is a limit point of $A + B$, while it's not in $A + B$. It follows that $A + B$ is not closed.

Problem 20

(a) We will prove it by means of ternary expansion.

Proof. We first prove $[0,1] \subset A + B$. Assume $x \in [0,1]$. As noted, let C be the Cantor set, $A = C$, and B $= \frac{C}{2}$. Then A consists of all numbers which have a ternary expansion **with only 0 and 2 in all places of the ternary representation**. This implies that B consists of all numbers which have a ternary expansion **with only 0 and 1 in all places of the ternary representation**. Now any number $x \in [0,1]$ can be written as $a + b$ where $a \in A$ and $b \in B$ as follows:
Pick any ternary expansion $0.x_1 x_2 \ldots$ for x. Define

$$a_n = \begin{cases} x_n & \text{if } x_n = 2 \\ 0 & \text{if else} \end{cases}, \tag{2}$$

and

$$b_n = \begin{cases} x_n & \text{if } x_n = 1 \\ 0 & \text{if else} \end{cases}. \tag{3}$$

Then we have $a = 0.a_1 a_2 \ldots$ and $b = 0.b_1 b_2 \ldots$, which follows that $x = a + b$. □

(b) It's clear that

$$A + B = \{(a,0) + (0,b) : a, b \in [0,1]\} = \{(a,b) : a, b \in I\} = I \times I.$$

Examples in (a) and (b) are exactly those $m(A) = m(B) = 0$ where $m(A + B) > 0$.

Problem 21

Proof. As shown in exercise 2, there is a continuous function $f : [0,1] \to [0,1]$ such that $f(C) = [0,1]$, where C is the Cantor set.
Let $N \subset [0,1]$ be the non-measurable set constructed in theorem 3.6 of the book. Let

$$E = f^{-1}(N) \bigcap C,$$

we have $m(E) \leq m(C) = 0$ and $f(N) = E$, which completes the proof. □

Problem 22

Proof. We will prove it by contradiction. Suppose that such an f exists. Since $f(x) = \chi_{[0,1]}(x) = x$ for a.e. $x \in \mathbb{R}$, $\exists$ $x_1 \in (0,1)$ and $x_2 \in (1,2)$ such that $f(x_1) = 1$ and $f(x_2) = 0$.
Since f is continuous, $\exists$ $x_3 \in (x_1, x_2)$ such that $f(x_3) = \frac{1}{2}$. It follows that

$$\exists \delta > 0, \forall x \in O(x_3, \delta), |f(x) - \frac{1}{2}| < \frac{1}{4}.$$

Then it's clear that

$$f(x) \neq \chi_{[0,1]}(x), x \in O(x_3, \delta),$$

and $m(O(x_3, \delta)) = 2\delta > 0$.
Contradict the condition that $f(x) = \chi_{[0,1]}(x) = x$ for a.e. $x \in \mathbb{R}$! □

Problem 23

Proof. Let $n \in N^*$ and $x \in \mathbb{R}$.Define

$$B_n(x) = \frac{k}{2^n}, if \frac{k}{2^n} \leq x < \frac{k+1}{2^n}.$$

Then we let $f_n(x,y) = f(B_n(x), y)$. If f is the pointwise limit of measurable functions f_n, it is measurable.
Next we are going to show that $f_n \to f$ for every x and f_n is measurable for every n.
First it's clear that $f_n \to f$ for every x. (usual procedure of exmaining)
Second We are to prove f_n is measurable. We have for every $a \in \mathbb{R}$:

$$\begin{aligned}\{(x,y) : f_n(x,y) > a\} &= \bigcup_{k=-\infty}^{\infty} \{(x,y) : \frac{k}{2^n} \leq x < \frac{k+1}{2^n}, f(\frac{k}{2^n}, y) > a\} \\ &= \bigcup_{k=-\infty}^{\infty} [\frac{k}{2^n}, \frac{k+1}{2^n}) \times \{y : f(\frac{k}{2^n}, y) > a\},\end{aligned}$$

which follows that f_n is measurable.

□

Problem 24

Sketch: We will **below** find an enumeration $\{r_n\}_{n=1}^{\infty}$where the only rationals outside a fixed bounded interval, denoted bt I, take the form r_{m^2} where m is some integral. Thus we have

$$\sum_{r_{m^2}\notin I} m\Big((r_{m^2}-\frac{1}{m^2}, r_{m^2}+\frac{1}{m^2})\Big) \leq \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

, which follows that the open intervals centered by the rationals outside can't fill the complement $\mathbb{R}-I$. In other words we have

$$\bigcup_{n=1}^{\infty}\left(r_n-\frac{1}{n}, r_n+\frac{1}{n}\right) \neq \mathbb{R}.$$

Proof. We will construct an enumeration. Suppose I=[0,1]. Since $Q\bigcap I$ is countable, we have an enumeration

$$Q\bigcap I = \{r_i\}_{i=1}^{\infty}.$$

Similarly we have an enumeration

$$Q\bigcap(\mathbb{R}-I) = \{s_i\}_{i=1}^{\infty}.$$

Then we construct an eneuration below

$$t_j = \begin{cases} r_{j-[\sqrt{j}]} & \text{if } j\neq m^2 \\ s_m & \text{if } j = m^2 \end{cases}, \tag{4}$$

which immediately follows that

$$Q = \{t_i\}_{i=1}^{\infty}.$$

And in the enumeration the only rationals outside I, take the form r_{m^2} where m is some integral. We have

$$\begin{aligned} m\Big(\bigcup_{n=1}^{\infty}(t_n-\frac{1}{n}, t_n+\frac{1}{n})\Big) &= m\Big(\bigcup_{j=1}^{\infty}(r_{j-[\sqrt{j}]}-\frac{1}{j}, r_{j-[\sqrt{j}]}+\frac{1}{j}) + \bigcup_{m=1}^{\infty}(s_m-\frac{1}{m^2}, s_m+\frac{1}{m^2})\Big) \\ &\leq m\Big(\bigcup_{j=1}^{\infty}(r_{j-[\sqrt{j}]}-\frac{1}{j}, r_{j-[\sqrt{j}]}+\frac{1}{j})\Big) + m\Big(\bigcup_{m=1}^{\infty}(s_m-\frac{1}{m^2}, s_m+\frac{1}{m^2})\Big) \\ &\leq 3+\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty, \end{aligned}$$

which follows from $m(R)=\infty$ that

$$\Big(\bigcup_{n=1}^{\infty}(r_n-\frac{1}{n}, r_n+\frac{1}{n})\Big)^c \neq \emptyset.$$

□

Problem 25

Proof. Assume $\epsilon > 0$ and E is measurable. It's clear that E^c is measurable, which follows that there is an open set O containing E^c with

$$m^*(O-E^c) < \epsilon.$$

Then we have $O^c \subset E$ where O^c is closed and

$$m^*(E-O^c) = m^*(O-E^c) < \epsilon.$$

Problem 25 continued on next page...

Let $F = O^c$ and we prove one direction. Conversely assume that for every $\epsilon > 0$ there is a closed set F contained in E with $m^*(E - F) < \epsilon$. Then we have

$$m^*(F^c - E^c) = m^*(E - F) < \epsilon.$$

Let $O = F^c$ and we have the assertion that for every $\epsilon > 0$ there is a open set O containing E^c with $m^*(O - E^c) < \epsilon$, which follows that E^c is measurable and thus E is measurable. □

Problem 26

Proof. Assume $\epsilon > 0$. Since B is measurable, there exist an open set O containing B with $m^*(O - B) < \epsilon$. And since $B - A$ is measurable, we have the equation

$$m^*(B - A) = m(B - A) = m(B) - m(A) = 0.$$

It follows that

$$m^*(O - E) = m^*\Big((O - B) + (B - E)\Big) \le m^*(O - B) + m^*(B - E)$$
$$\le m^*(O - B) + m^*(B - A) = m^*(O - B) < \epsilon,$$

which follows that E is measurable. □

Problem 27

Proof. Define $E_t^{d+1} = \{x \in \mathbb{R}^d : |x| \le t\}$. Then we define

$$f(t) = m\Big(((E_2 - E_1)\bigcap E_t^{d+1})\bigcup E_1\Big), 0 \le t < \infty.$$

Next what we need to do is to study the properties of $f(t)$.
First since it's clear that $f(t)$ is **bounded**, there exist $N > 0$ such that

$$E_2 \subset E_N^{d+1}.$$

Then we have these conditions for N above:

$$a \le f(t) \le b, 0 \le t \le N, f(0) = m(E_1), f(N) = m(E_2).$$

Second $f(t)$ is **continuous**. Asssume $x_0 \in [0, \infty)$ and $\epsilon > 0$. Since $g(t) := \frac{4\pi t^3}{3}$ is continuous at x_0, for ϵ, there exist $\delta_0 > 0$ such that

$$|y - x| < \delta_0 \implies |\frac{4\pi x^3}{3} - \frac{4\pi y^3}{3}| < \epsilon.$$

Then $\exists \delta = \delta_0$ such that

$$|y - x| < \delta \implies |f(y) - f(x)| \le m(E_y^{d+1} - E_x^{d+1}) = |\frac{4\pi x^3}{3} - \frac{4\pi y^3}{3}| < \epsilon.$$

Then since f is continuous and bounded, f sends the connected set [0,N] to $[a, b]$ in $\mathbb{R}$. Thus we have proved that for any c with a<c<b, there is a number t with $0 \le t \le N$ such that $f(t) = c$. In other words, there is a compact set

$$E_1 \subset E = \Big(((E_2 - E_1)\bigcap E_t^{d+1})\bigcup E_1\Big) \subset E_2$$

such that $m(E) = c$. □

Problem 28

Proof. Assume $\alpha > 0$. We let $\epsilon = (\frac{1}{\alpha} - 1) * m^*(E) > 0$. Since

$$m^*(E) = \underset{E\subset O(openset)}{inf} \{m^*(O)\} > 0,$$

there exist an open set O such that

$$m^*(O) \leq m^*(E) + \epsilon = \frac{m^*(E)}{\alpha}.$$

Write O as a countable union of disjoint open intervals, as follows:

$$O = \bigcup_{t\in T} I_t, \text{where the open intervals } I_t\text{'s are disjoint.} \tag{5}$$

Therefore $\exists I_y \in \{I_t : t \in T\}$ s.t.

$$m^*(E \bigcap I_y) \geq \alpha m^*(I_y).$$

Otherwise

$$m^*(E \bigcap I_t) < \alpha m^*(I_t), \text{ for every t} \in \text{T},$$

which follows that

$$m^*(E) = \sum_{t\in T} m^*(E \bigcap I_t) < \alpha \sum_{t\in T} m^*(I_t) = \alpha m^*(O).$$

A contradiction! □

Problem 29

Proof. From exercise 28, there exist an open interval I such that

$$m(E \bigcap I) \geq \frac{9}{10} m(I)$$

.
Denote $E \bigcap I = E_0$. Arbitrarily suppose $0 < |\alpha| < \frac{m(I)}{10}$. We first prove E and $(E + \alpha)$ must have nonempty intersection for every $0 < |\alpha| < \frac{m(I)}{10}$.
Let $\bar{I} = I \bigcap (I + \alpha)$ and $m(\bar{I})$ is greater than $\frac{9}{10} m(I)$. Since $m(E \bigcap I) \geq \frac{9}{10} m(I)$ and **the translation invariance of Lebesgue measure**,

$$m\Big((E + \alpha) \bigcap (I + \alpha)\Big) \geq \frac{9}{10} m(I).$$

Now we have

$$\frac{9}{10} m(I) \leq m(E \bigcap I) = m(E \bigcap \bar{I}) + m(E \bigcap (I - \bar{I})) \leq m(E \bigcap \bar{I}) + m(I - \bar{I}) = m(E \bigcap \bar{I}) + \alpha,$$

which follows that

$$m(E \bigcap \bar{I}) \geq (\frac{9}{10} - \frac{1}{10}) m(I) = \frac{4}{5} m(I).$$

Similarly, $m((E + \alpha) \bigcap I_0) > \frac{4}{5} m(I)$. Now if $(E + \alpha)$ and E were disjoint, this would imply

$$m(\bar{I}) \geq m(E \bigcap \bar{I}) + m((E + \alpha) \bigcap \bar{I}) \geq \frac{8}{5} m(I).$$

Problem 29 continued on next page...

But $m(\bar{I}) = m(I) - \alpha < m(I)$. So E and $(E+\alpha)$ must have nonempty intersection for every $0 < |\alpha| < \frac{m(I)}{10}$. Let $x \in E\bigcap(E+\alpha)$. Then $\exists e_1, e_2 \in E$ such that

$$e_1 = x = e_2 + \alpha \implies e_1 - e_2 = \alpha.$$

Hence $\alpha \in E - E$ for every $\alpha \in (-\frac{m(I)}{10}, \frac{m(I)}{10})$, which implies that $E - E$ contains an open interval around the origin. □

Problem 30

It's just a generalization of problem 29.

Proof. By exercise 28, there exist open intervals I_1 and I_2 such that

$$m(E\bigcap I_1) \geq \frac{9}{10}m(I_1) \quad \text{and} \quad m((F\bigcap I_2) \geq \frac{9}{10}m(I_2).$$

WLOG assume $m(I_2) \leq m(I_1)$. Then $\exists t_0 \in \mathbb{R}$ such that

$$I_2 + t_0 \subset I_1.$$

We first prove E and $(F + t_0 + \alpha)$ must have nonempty intersection for every $0 < |\alpha| < \frac{m(I_2)}{10}$. For every $0 < |\alpha| < \frac{m(I_2)}{10}$, we have

$$m\bigg((I_2 + t_0 + \alpha)\bigcap I_1\bigg) \geq \frac{9}{10}m(I_2).$$

Let $\bar{I} = I_1\bigcap(I_2+t_0+\alpha)$ and $m(\bar{I})$ is greater than $\frac{9}{10}m(I)$. Since $m(F\bigcap I_2) \geq \frac{9}{10}m(I_2)$ and **the translation invariance of Lebesgue measure**,

$$m\bigg((F + t_0 + \alpha)\bigcap(I_2 + t_0 + \alpha)\bigg) \geq \frac{9}{10}m(I_2).$$

Then we have

$$\frac{9}{10}m(I_1) \leq m(E\bigcap I_1) = m(E\bigcap \bar{I}) + m(E\bigcap(I_1 - \bar{I})) \leq m(E\bigcap \bar{I}) + m(I_1) - \frac{9}{10}m(I_2),$$

which follows that

$$m(E\bigcap \bar{I}) \geq \frac{9}{10}m(I_2) - \frac{1}{10}m(I_1).$$

Similarly we have

$$\begin{aligned}
\frac{9}{10}m(I_2) &\leq m\bigg((F + t_0 + \alpha)\bigcap(I_2 + t_0 + \alpha)\bigg) \\
&= m\bigg((F + t_0 + \alpha)\bigcap \bar{I}\bigg) + m\bigg((F + t_0 + \alpha)\bigcap\bigg((I_2 + t_0 + \alpha) - \bar{I}\bigg)\bigg) \\
&\leq m\bigg((F + t_0 + \alpha)\bigcap \bar{I}\bigg) + \frac{1}{10}m(I_2),
\end{aligned}$$

which follows that

$$m\bigg((F + t_0 + \alpha)\bigcap \bar{I}\bigg) \geq \frac{4}{5}m(I_2).$$

Problem 30 continued on next page...

If $E\bigcap(F+t_0+\alpha)\neq\emptyset$, we have

$$\begin{aligned} m(\bar{I}) &\geq m(E\bigcap\bar{I})+m((F+t_0+\alpha)\bigcap\bar{I}) \\ &\geq \frac{9}{10}m(I_2)-\frac{1}{10}m(I_1)+\frac{4}{5}m(I_2) \\ &= \frac{4}{5}(m(I_1)+m(I_2))\geq\frac{8}{5}m(I_2). \end{aligned}$$

A contradiction! Thus we have E and $(F+t_0+\alpha)$ must have nonempty intersection for every $0<|\alpha|<\frac{m(I_2)}{10}$. Similarly we have $\alpha\in E-F$ for every $\alpha\in(-\frac{m(I_2)}{10},\frac{m(I_2)}{10})$, which implies that $E-F$ contains an open interval around the origin. □

Problem 31

Proof. Suppose N^* is measurable. Given $\mathbb{Q}$ an enuermeration $\{r_n\}$. If N^* is measurable, then so are its translations $N^*_n=N^*+r_n$.
First it's clear that $\bigcup_{n=1}^{\infty}N^*_n=\mathbb{R}$ and $\{N^*_n\}_{n=1}^{\infty}$ are disjoint, which follows that

$$\sum_{n=1}^{\infty}m(N^*_n)=m(\bigcup_{n=1}^{\infty}N^*_n)=m(\mathbb{R})=\infty$$

Since the translation invariance of Lebesgue measure, $m(N^*_n)=m(N^*)$. It follows from the fomula above that $m(N^*)>0$. By problem 29, (N^*-N^*) contains an open interval around the origin. WLOG, we suppose the open interval $(-r,r)$ where r>0.
We have

$$\forall|x|<r,\exists n_1,n_2\in N^*\text{s.t.}(x=n_1-n_2).$$

However, if x is a rational number s.t. $0<|x|<r$, then from $x=n_1-n_2$ we have $n_1\ n_2$, which follows that $n_1=n_2$. Then x=0, a contradiction! Thus we have proved N^* is non-measurable. □

Problem 32

(a) Consider the translates of E by rationals.

Proof. Give an enumeration of $[0,1]\bigcap\mathbb{Q}=\{r_n:n=1,2,\ldots\}$. Since $\{E+r_n:n=1,2,\ldots\}$ are disjoint, we have

$$\bigcup_{n=1}^{\infty}(E+r_n)\subset[-1,2]\implies\sum_{n=1}^{\infty}m(E+r_n)\leq m([-1,2])=3\implies m(E)=0.$$

□

(b) Proof is direct similarly.

Proof. Let $A\subset\mathbb{R}$ be a set with positive outer measure. Let $N_r=N+r$ be the Vitali set (non-measurable set constructed in Chapter 1)on the real line translated by r. For r,q rational, $r\neq q$, we have $N_r\cap N_q$ is empty, and

$$\bigcup_{r\in\mathbb{Q}}N_r=\mathbb{R}.$$

So $A=\bigcup_r(A\bigcap E_r)$, and

$$m^*(A)\leq\sum_r m^*(A\cap E_r)$$

.

Now if $A \bigcap N_r$ is measurable, then it must have measure 0 by the preceding paragraph since its set of differences contains no interval at the origin (any two elements of this set differ by an irrational number). But since $m^*(A) > 0$, we must have $A \bigcap N_r$ with positive outer measure for some r. Then the $A \bigcap N_r$ is the desired nonmeasurable subset. □

Plus: This proposition can be extended to $\mathbb{R}^n$ space similarly. We just omit it.

Problem 33

Proof. First we prove $m^*(N^c) = 1$ by contradiction.
Suppose $m^*(N^c) = \alpha < 1$. Then since

$$m^*(N^c) = \underset{N^c \subset O(open set)}{inf} \{m^*(O)\},$$

there exist an open set O containing N^c such that

$$m^*(O) - m^*(N^c) < \frac{1-\alpha}{2}.$$

We let $U = O \bigcap I$ which is clearly measurable. And we have $N^c \subset U$ and $U \subset I$ and

$$m^*(U) - m^*(N^c) \leq m^*(O) - m^*(N^c) < \frac{1-\alpha}{2}.$$

Hence we have $U^c \subset N$ and U^c is clearly measurable, which contradicts the conclusion in problem 32(a).
Second we prove $m^*(E_1) + m^*(E_2) \neq m^*(E_1 + E_2)$. We have

$$m^*(E_1 + E_2) = m([0,1]) = 1 \quad and \quad m^*(E_1) + m^*(E_2) > m^*(E_2) = 1.$$

It's clear from above the formula holds. □

Problem 34

Proof. Any Cantor set can be put in **bijective correspondence with the set of 0-1 sequences** as follows:
Given $x \in C$, where $C = \bigcap_{i=1}^{\infty} C_i$ is a Cantor set.
1) Define $x_1 = 0$ if x is in the left of the two intervals in C_1 (call this left interval I_0), and $x_1 = 1$ if x is in the right interval I_1.
2) Define $x_2 = 0$ if x is in the left subinterval (either I_{00} or I_{10}) in C_2, and $x_2 = 1$ if x is in the right subinterval.
3) Continuing the procedure inductively.

Continuing in this fashion we obtain a **bijection** from C to the set of 0-1 sequences, denoted by Φ_C.

Note that this bijection is **monotonically increasing** in the sense that for $x, y \in C$, if $y > x$ then $y_n > x_n$ at the first point n in the sequence at which x_n and y_n differ. Now we can create an increasing bijection f=$\Phi_{C_2}^{-1} \circ \Phi_{C_1}$ from C_1 to C_2 by the composition of the mapping Φ_{C_1} from C_1 to 0-1 sequences and the mapping $\Phi_{C_2}^{-1}$ from 0-1 sequences to C_2.

This function f is **continuous on** C_1. Because if $x, y \in C_1$ are close, their corresponding sequences $\{x_n\}$ and $\{y_n\}$ will agree in their first N terms; then $f(x)$ and $f(y)$ will agree in their first N terms as well, which means they're in the same subinterval of the N-th iterate of C_2, which has length at most $\frac{1}{2}N$. Hence $f(x)$ and $f(y)$ can be made arbitrarily close if x and y are sufficiently close.

Then since C_1 is compact, we can **extend** f to a continuous bijection on all of [0, 1] in a piecewise linear fashion similar to the construction in problem 18, denoted by F.
Since $[0,1]-C_1$ is open in $\mathbb{R}$, we have $[0,1]-C_1$ can be written uniquely as countable union of disjoint open intervals, say

$$[0,1]-C_1 = \bigcup_{x\in([0,1]-C_1)} I_x.$$

Then we define

$$F(x) = \begin{cases} f(x) & \text{if } x \in C_1 \\ f(a) + \frac{x-a}{b-a} * b & \text{if } I_x = (a,b) \end{cases}. \tag{6}$$

It's clear that $F : [0,1] \to [0,1]$ is continuous on [0,1] and monotonically increasing.

This construction will also clearly preserve the bijectivity of f. Hence we have a continuous bijection $f : [0,1] \to [0,1]$ with $f(C_1) = C_2$, which clearly satisfy (i), (ii) and (iii) proved above. □

Problem 35

Proof. Let $\Phi : C_1 \to C_2$ as in exercise 34, with $m(C_1) > 0$ and $m(C_2) = 0$. By the conclusion in problem 32(b), since $m(C_1) > 0$, $\exists N \subset C_1$ such that N is non-meaurable.
Take $f = \chi_{\Phi(N)}$, then it's clear that f is measurable (f is simple) and Φ is continuous. However,

$$f \circ \Phi(x) = \begin{cases} \Phi(x) & \text{if } x \in N \\ 0 & \text{if else} \end{cases}. \tag{7}$$

Therefore we have $f \circ \Phi$ is non-measurable since $\{f < 2\} = N$ is non-measurable.
Then since $\Phi(x)$ is a monnotonically increasing continuous function from $[0,1] \to [0,1]$, which implies that for every $y \in [0,1]$, there exists a unique $x \in [0,1]$, such that $y = \Phi(x)$. Thus Φ and Φ^{-1} maps Borel sets into Borel sets in $[0,1]$.
Now choose a non-Borel subset $S \subset C_1$. Its image $\Phi(S)$ must be Lebesgue measurable, as a subset of C_2, but it is not Borel measurable! □

Problem 36

(a) Just follow the hint.

Proof. **First I claim that we can construct a closed nowhere dense set of positive measure inside a given interval in** $\mathbb{R}$ (i.e.: If $J \subset \mathbb{R}$ is any nonempty interval, there exists a closed set $S \subset I$ such that S has positive Lebesgue measure and S has empty interior). For instance we may construct a "Cantor-like set" as in exercise 4.

If $[0,1] \subset \mathbb{R}$ is any nonempty interval, there exists a closed set $S \subset [0,1]$ such that S has positive Lebesgue measure and S has empty interior. Doing the same construction inside any **open interval** $J \subset [0,1]-S$ we may find another closed set, denoted by T, with positive measure and empty interior such that $S \cap T = \emptyset$.

Now the set of **rational intervals** (that is, intervals with rational center and rational radius) is countable. Thus we may enumerate them: $I_1, I_2, I_3, ...$ Now, as above, we may find a pair of closed disjoint positive-measure empty-interior sets

$$S_1, T_1 \subset I_1.$$

Since S_1 and T_1 have empty interior, $S_1 \bigcup T_1$ has empty interior by the **Baire category theorem**. Now since S_1, T_1 are closed and their union cannot contain I_2, $I_2 - S_1 - T_1$ contains a non-empty open interval J_2.

By working inside J_2, we may find closed sets $S_2, T_2 \subset I_2$ with positive measure and empty interior and such that S_1, T_1, S_2, T_2 are pairwise disjoint. Then working inside an open interval in

$$I_3 - S_1 - T_1 - S_2 - T_2$$

we may find closed sets $S_3, T_3 \subset I_3$ with positive measure and empty interior and such that $S_1, T_1, S_2, T_2, S_3, T_3$ are pairwise disjoint. Continuing in this way we get a sequence of pairwise disjoint sets with positive measure

$$S_1, T_1, S_2, T_2, S_3, T_3, ...$$

such that every rational interval contains one (maybe infinitely many) of the pairs S_j, T_j.

Now define the set $E := \bigcup_{k=1}^{\infty} S_k$ and let K be an arbitrary nonempty open interval of positive length. Then K contains a rational interval, and therefore K contains a pair S_j, T_j. Then since $E \cap T_j = \emptyset$ and $S_j \subset E$, we have

$$T_j \subset K - E \subset K - S_j.$$

Therefore for any open sub-interval $K \subset [0, 1]$:

$$m(K \bigcap E) \geq m(S_j) > 0 \quad \text{and} \quad m(K \bigcap E^c) = m(K - E) \geq m(T_j) > 0.$$

We have proved that E possesses the property.

□

(b) The proof is direct from conclusion in (a).

Proof. First suppose $g(x) = f(x)$ a.e. x.
If g is continuous at some point $x_0 \in E - \{0, 1\}$. Assume $0 < \epsilon < 1$, then $\exists \delta > 0$ such that

$$|y - x_0| < \delta \implies |g(y) - g(x_0)| < \epsilon < 1.$$

It's clear that $f(x_0) = 1$. We consider some arbitrarily small open sub-interval containing x_0, denoted by K, where $0 < m(K) < \delta$. Since $g(x) = f(x)$ a.e. x and $m(E^c \bigcap K) > 0$, there exist some $\bar{y} \subset E^c \bigcap K$ such that $g(\bar{y}) = 1$, then

$$|\bar{y} - x_0| < \delta \quad \text{and} \quad |g(\bar{y}) - g(x_0)| = 1.$$

A contradiction! Thus g can't be continuous at some point $x_0 \in E - \{0, 1\}$. Simlarly we can prove the case when $x_0 \in E^c - \{0, 1\}$ and when $x_0 = 0, 1$. Finally we prove problem 36! □

Problem 37

Suppose a function $F(x, y) = (x, f(x))$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose image is the curve Γ.

Proof. First it's clear that $\mathbb{R} \times \{0\}$ is a zero-measure set in $\mathbb{R} \times (-1, 1) \subset \mathbb{R}^2$.
Second we prove that $F(\mathbb{R} \times \{0\})$ is a of zero measure set **of zero measure** in $\mathbb{R}^2$. It's clear that F is continuous on $\mathbb{R} \times (-1, 1)$. Moreover, since $\mathbb{R} \times (-1, 1)$ is open, $\mathbb{R} \times (-1, 1)$ can be written as a countable union of closed rectangles, written as

$$\mathbb{R} \times (-1, 1) = \bigcup_{i=1}^{\infty} I_i \quad \text{where } I_i \text{ are closed rectangles in } \mathbb{R}^2.$$

Then we have

$$\mathbb{R} \times \{0\} = \bigcup_{i=1}^{\infty} \left((\mathbb{R} \times \{0\}) \bigcap I_i \right)$$

Since Lipschitz continuity holds in every I_i for F(x,y). Similar to proof of problem 8, we have $F\Big((\mathbb{R} \times \{0\}) \bigcap I_i)\Big)$ of zero measure for every i. And it's clear that

$$\begin{aligned} m\Big(F(\mathbb{R} \times \{0\})\Big) &= m\left(F\left(\bigcup_{i=1}^{\infty} \left((\mathbb{R} \times \{0\}) \bigcap I_i\right)\right)\right) \\ &= m\left(\bigcup_{i=1}^{\infty} F\Big((\mathbb{R} \times \{0\}) \bigcap I_i)\Big)\right) \\ &\le \sum_{i=1}^{\infty} m\left(F\Big((\mathbb{R} \times \{0\}) \bigcap I_i\Big)\right) = 0, \end{aligned}$$

which follows that Γ is of zero measure.

□

Problem 38

We will omit it since it's **usual process of fundamental analysis**.

Problem 39

Proof. We will prove it with the following the steps.
1) Base step:

$$P(2) : (\sqrt{x_1} - \sqrt{x_2})^2 \ge 0 \implies x_1 + x_2 - 2\sqrt{x_1 x_2} \ge 0 \implies x_1 + x_2 \ge 2\sqrt{x_1 x_2} \implies \frac{x_1 + x_2}{2} \ge \sqrt{x_1 x_2}$$

$P(2)$ is true.
2) Inductive Step: (composed of Forward Part a) and Backward Part b))
a) Forward Part:
Assume the inequality holds for $n = k$, ie. $P(k)$ is true.

$$P(k) : \frac{x_1 + x_2 + ... + x_k}{k} \ge \sqrt[k]{x_1 . x_2 .... x_k}$$

Problem 39 continued on next page...

Then we have

$$P(2k) : \frac{x_1 + x_2 + ... + x_{2k}}{2k} = \frac{1}{k}\Big(\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2} + ... + \frac{x_{2k-1+x_{2k}}}{2}\Big)$$
$$\geq \frac{\sqrt{x_1x_2} + \sqrt{x_3x_4} + ... + \sqrt{x_{2k-1}x_{2k}}}{k} \geq \sqrt[k]{\sqrt{x_1x_2x_3....x_{2k-1}x_{2k}}} = \sqrt[2k]{x_1x_2...x_{2k}}$$

Thus $P(2k)$ is true whenever $P(k)$ is true.
b) Backward Part:
Assume the inequality holds for $n = k$, ie. $P(k)$ is true.

$$P(k) : \frac{x_1 + x_2 + ... + x_k}{k} \geq \sqrt[k]{x_1.x_2....x_k}$$

Then we have

$$P(k-1) : \frac{x_1 + x_2 + ... + x_{k-1} + \frac{x_1+x_2+...+x_{k-1}}{k-1}}{k} \geq \sqrt[k]{x_1x_2...x_{k-1}.\frac{x_1x_2...x_{k-1}}{k-1}}$$
$$\Leftrightarrow \left(\frac{x_1 + x_2 + ... + x_{k-1}}{k-1}\right)^k \geq x_1x_2...x_{k-1}.\frac{x_1x_2...x_{k-1}}{k-1}$$
$$\Leftrightarrow \left(\frac{x_1 + x_2 + ... + x_{k-1}}{k-1}\right)^{k-1} \geq \frac{x_1x_2...x_{k-1}}{k-1}$$
$$\Leftrightarrow \frac{x_1 + x_2 + ... + x_{k-1}}{k-1} \geq \sqrt[k-1]{\frac{x_1x_2...x_{k-1}}{k-1}}$$

Thus $P(k-1)$ is true whenever $P(k)$ is true.
Finally by the forward-backward induction AM-GM inequality is true for every n. □