RAhw20260525

习题 1 (Stein,Ch3,T16) Show that if F is of bounded variation in $[a,b]$, then:

(a) $\displaystyle\int_a^b |F'(x)|\,\mathrm{d}x \le T_F(a,b)$.

(b) $\displaystyle\int_a^b |F'(x)|\,\mathrm{d}x = T_F(a,b)$ if and only if F is absolutely continuous.

As a result of (b), the formula $L = \displaystyle\int_a^b |z'(t)|\,\mathrm{d}t$ for the length of a rectifiable curve parametrized by z holds if and only if z is absolutely continuous.

习题 2 (Stein,Ch3,T19) Show that if $f:\mathbb{R}\to\mathbb{R}$ is absolutely continuous, then

(a) f maps sets of measure zero to sets of measure zero.

(b) f maps measurable sets to measurable sets.

提示：对于 (a)，考虑先证明：对 $\forall\varepsilon>0, \exists\delta>0, \text{s.t.} \ \forall \sum\limits_{n=1}^{\infty}|b_n-a_n|<\delta$，有 $\sum\limits_{n=1}^{\infty}|f(b_n)-f(a_n)|<\varepsilon$，其中 $I_n=(a_n,b_n)$ 是一族两两不交的区间；再使用开集结构定理

对于 (b)，先用连续性证明对开集 O，$f(O)$ 可测，再将可测集写为 G_δ 集与零测集的差集

习题 3 (Stein,Ch3,T20) This exercise deals with functions F that are absolutely continuous on $[a,b]$ and are increasing. Let $A=F(a)$ and $B=F(b)$.

(a) There exists such an F that is in addition strictly increasing, but such that $F'(x)=0$ on a set of positive measure.

(b) The F in (a) can be chosen so that there is a measurable subset $E\subset[A,B]$, $m(E)=0$, so that $F^{-1}(E)$ is not measurable.

(c) Prove, however, that for any increasing absolutely continuous F, and E a measurable subset of $[A,B]$, the set $F^{-1}(E)\cap\{F'(x)>0\}$ is measurable.

Hint: (a) Let $F(x)=\int_a^x \chi_K(t)\,dt$, where K is the complement of a Cantor-like set C of positive measure. For (b), note that $F(C)$ is a set of measure zero. Finally, for (c) prove first that $m(O)=\int_{F^{-1}(O)} F'(x)\,dx$ for any open set O.

习题 4 (Stein,Ch3,T32) Let $f:\mathbb{R}\to\mathbb{R}$. Prove that f satisfies the Lipschitz condition

$$|f(x)-f(y)|\le M|x-y|$$

for some M and all $x,y\in\mathbb{R}$, if and only if f satisfies the following two properties:

(i) f is absolutely continuous.

(ii) $|f'(x)|\le M$ for a.e. x.