RAhw20260429

习题 1 (Stein,Ch2,T14) In Exercise 6 of the previous chapter we saw that $m(B) = v_d r^d$, whenever B is a ball of radius r in $\mathbb{R}^d$ and $v_d = m(B_1)$, with B_1 the unit ball. Here we evaluate the constant v_d.

(a) For $d = 2$, prove using Corollary 3.8 that

$$v_2 = 2\int_{-1}^{1} (1 - x^2)^{1/2}\, dx,$$

and hence by elementary calculus, that $v_2 = \pi$.

(b) By similar methods, show that

$$v_d = 2v_{d-1}\int_{0}^{1} (1 - x^2)^{(d-1)/2}\, dx.$$

(c) The result is

$$v_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}.$$

Another derivation is in Exercise 5 in Chapter 6 below. Relevant facts about the gamma and beta functions can be found in Chapter 6 of Book II.

习题 2 (Stein,Ch2,T17) Suppose f is defined on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} a_n, & n \le x < n+1,\ n \le y < n+1,\ n \ge 0, \\ -a_n, & n \le x < n+1,\ n+1 \le y < n+2,\ n \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $a_n = \sum_{k \le n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

(a) Verify that each slice f^y and f_x is integrable. Also for all x, $\int f_x(y)\, dy = 0$, and hence

$$\int \left(\int f(x,y)\, dy \right) dx = 0$$

.

(b) However, $\int f^y(x)\, dx = a_0$ if $0 \le y < 1$, and $\int f^y(x)\, dx = a_n - a_{n-1}$ if $n \le y < n+1$ with $n \ge 1$. Hence $y \mapsto \int f^y(x)\, dx$ is integrable on $(0, \infty)$ and

$$\int \left(\int f(x,y)\, dx \right) dy = s$$

(c) Note that

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\, dx\, dy = \infty$$

习题 3 (Stein,Ch2,T20) The problem (highlighted in the discussion preceding Fubini's theorem) that certain slices of measurable sets can be non-measurable may be avoided by restricting attention to Borel measurable functions and Borel sets. In fact, prove the following:

Suppose E is a Borel set in $\mathbb{R}^2$. Then for every y, the slice E^y is a Borel set in $\mathbb{R}$.

Hint: Consider the collection $\mathcal{C}$ of subsets E of $\mathbb{R}^2$ with the property that each slice E^y is a Borel set in $\mathbb{R}$. Verify that $\mathcal{C}$ is a σ-algebra that contains the open sets.

习题 4 Prove that $\forall f \in L^p$

$$||f||_p^p = p\int_0^{\infty} m(\{|f| > t\})t^{p-1}\mathrm{d}t$$