注：在做题过程中若使用控制收敛定理 (DCT) 时应指出 (可积的) 控制函数

习题 1 设 $f(x)=\frac{\sin x}{x}, x\in(-\infty,+\infty)$，证明 $f\notin L^1(\mathbb{R})$

提示：由对称性只需考虑正实轴上的积分，注意到

$$[0,+\infty)=\bigcup_{n=0}^{\infty}[n\pi,(n+1)\pi)\stackrel{\text{def}}{=}\bigcup_{n=0}^{\infty}I_n$$

再对 $|f|=\sum\limits_{n=0}^{\infty}|f|\chi_{I_n}$ 使用逐项积分定理，并在 I_n 上对 $\frac{1}{x}$ 放缩

习题 2 计算

$$\lim_{n\to\infty}\int_1^2\frac{n^2\sin\left(\frac{x}{n}\right)}{1+nx^2}\mathrm{d}x$$

习题 3 设 $E\in\mathscr{L}(\mathbb{R}^n), f\in L^1(E), f\geq 0$，证明

$$\lim_{n\to\infty}\int_E n\ln\left[1+\frac{f(x)}{n}\right]\mathrm{d}x=\int_E f\mathrm{d}m$$

习题 4（Stein,Ch2,T11） Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)\mathrm{d}x\geq 0$ for every measurable E, then $f(x)\geq 0$ a.e x. As a result, if $\int_E f(x)\mathrm{d}x=0$ for every measurable E, then $f(x)=0$ a.e

习题 5（Stein,Ch2,T12） Show that there are $f\in L^1(\mathbb{R}^d)$ and a sequence $\{f_n\}$ with $f_n\in L^1(\mathbb{R}^d)$ such that

$$||f-f_n||_{L^1}\to 0$$

but $f_n(x)\to f(x)$ for no x.

Hint: In $\mathbb{R}$, let $f_n=\chi_{I_n}$, where I_n is an appropriately chosen sequence of intervals with $m(I_n)\to 0$.

习题 6（Stein,Ch2,T15） Consider the function defined over $\mathbb{R}$ by

$$f(x)=\begin{cases}x^{-\frac{1}{2}}, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x)=\sum_{n=1}^{\infty}2^{-n}f(x-r_n)$$

Prove that F is integrable, hence the series defining F converges for almost every $x\in\mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F a.e is unbounded in any interval.