第二十九讲 (2026. 6. 10) ①

Def $(X_1 \times X_2, \mathcal{M}_1 \otimes \mathcal{M}_2, \mu_1 \times \mu_2)$

$E \subset X_1 \times X_2$

对 $x \in X_1$,

$$E_x \overset{\text{def}}{=} \{y \in X_2 : (x, y) \in E\}$$

称为 E 的 x-截面

对 $y \in X_2$,

$$E^y \overset{\text{def}}{=} \{x \in X_1 ; (x, y) \in E\}$$

称为 E 的 y-截面.

对 $X_1 \times X_2$ 上的函数 $f: X_1 \times X_2 \to \mathbb{R}$

$$f_x(y) \overset{\text{def}}{=} f(x, y), \quad y \in X_2$$

$$f^y(x) \overset{\text{def}}{=} f(x, y), \quad x \in X_1$$

分别称为 f 的 x-截面和 y-截面.

Prop

(i) 设 $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$，则

$$\begin{cases} \forall x \in X_1, & E_x \in \mathcal{M}_2 \\ \forall y \in X_2, & E^y \in \mathcal{M}_1 \end{cases} \qquad (*)$$

(ii) f 是 $\mathcal{M}_1 \otimes \mathcal{M}_2$-可测的

$$\Rightarrow \begin{cases} f_x & \mathcal{M}_2\text{-可测} \\ f^y & \mathcal{M}_1\text{-可测} \end{cases}$$

Pf (i) 设

$$\mathcal{F} \stackrel{def}{=} \{ E \subset X_1 \times X_2 : E \text{ 满足 } (*) \}$$

$\mathcal{F}$ 对可数并和差补封闭

$\Rightarrow \mathcal{F}$ 是 σ-代数.

而且 $\mathcal{M}_1 \times \mathcal{M}_2 \subset \mathcal{F}$

$\Rightarrow \mathcal{M}_1 \otimes \mathcal{M}_2 \subset \mathcal{F}$

(ii) $(f_x)^{-1}(B) = (f^{-1}(B))_x \in \mathcal{M}_2, \ \forall B \in \mathcal{B}_{\mathbb{R}}$

Thm (Tonelli)

设 $(X_1, \mathcal{M}_1, \mu_1)$
$(X_2, \mathcal{M}_2, \mu_2)$ 均 σ-有限

$f \in L^+(X_1 \times X_2)$

则

(T1) $$x \longmapsto \int_{X_2} f_x \, d\mu_2 \in L^+(X_1)$$

$$y \longmapsto \int_{X_1} f^y \, d\mu_1 \in L^+(X_2)$$

(T2)

$$\int_{X_1 \times X_2} f \, d(\mu_1 \times \mu_2) = \int_{X_1} \left[\int_{X_2} f(x,y) \, d\mu_2(y) \right] d\mu_1(x)$$

$$= \int_{X_2} \left[\int_{X_1} f(x,y) \, d\mu_1(x) \right] d\mu_2(y)$$

Thm (Fubini)

设 $(X_1, \mathcal{M}_1, \mu_1)$
$(X_2, \mathcal{M}_2, \mu_2)$ σ-有限.

$f \in L^1(X_1 \times X_2, \mu_1 \times \mu_2)$

$\boxed{?}$

(F1) $\begin{cases} f_x \in L^1(X_2, \mu_2) & \text{for } \mu_1\text{-a.e. } x \in X_1 \\ f^y \in L^1(X_1, \mu_1) & \text{for } \mu_2\text{-a.e. } y \in X_2 \end{cases}$

(F2) $\begin{cases} x \longmapsto \int_{X_2} f_x \, d\mu_2 \in L^1(X_1, \mu_1) \\ y \longmapsto \int_{X_1} f^y \, d\mu_1 \in L^1(X_2, \mu_2) \end{cases}$

(F3) $\boxed{3}$ (T2)

Pf of Fubini by Tonelli

$$\int_{X_1} \left[\int_{X_2} |f(x,y)| \, d\mu_2(y) \right] d\mu_1(x)$$

$$= \int_{X_1 \times X_2} |f| \, d(\mu_1 \times \mu_2) < +\infty$$

$$\Longrightarrow \quad x \longmapsto \int_{X_2} |f(x,y)| \, d\mu_2(y) \in L^1(\mu_1)$$

$\Rightarrow$ (F2) holds

$\Rightarrow x \mapsto \int_{X_2} |f_x| \, d\mu_2$ μ_1-a.e. 有限

$\Rightarrow$ 对 μ_1-a.e. $x \in X_2$. $f_x \in L^1(X_2, \mu_2)$

同理 对 μ_2-a.e. $y \in X_1$. $f^y \in L^1(X_1, \mu_1)$

$\Rightarrow$ (F1) holds

为证明 (F3)，对 f^+, f^- 应用 Tonelli

并相减 ((F2) 保证了可以相减)

Def 如果 $\mathcal{F} \subset 2^X$ s.t.

(i) $\mathcal{F} \ni E_k \nearrow E \Rightarrow E \in \mathcal{F}$

(ii) $\mathcal{F} \ni E_k \searrow E \Rightarrow E \in \mathcal{F}$

则称 $\mathcal{F}$ 是一个单调类

例: σ-代数是单调类.

Thm (单调类引理, Monotone Class Lemma) ⑥

设 $\mathcal{A}$ 是 X 上的代数. 则

$\mathcal{A}$ 生成的单调类 $=$ $\mathcal{A}$ 生成的 σ-代数

Pf 令

$$\mathcal{M} \overset{\text{def}}{=} \mathcal{A} \text{ 生成的 } \sigma\text{-代数}.$$

$$\mathcal{F} \overset{\text{def}}{=} \mathcal{A} \text{ 生成的单调类}$$

$\Rightarrow$ $\mathcal{F} \subset \mathcal{M}$ ($\because$ σ-代数是单调类)

$\Rightarrow$ 只需证:

Claim $\mathcal{F}$ 是 σ-代数.

对 $E \in \mathcal{F}$, 定义

$$\mathcal{F}_E \overset{\text{def}}{=} \{ F \in \mathcal{F} : E \setminus F,\ F \setminus E,\ E \cap F \in \mathcal{F} \}$$

Step 1 $\phi, E \in \mathcal{F}_E$

Step 2 $E \in \mathcal{F}_F \Leftrightarrow F \in \mathcal{F}_E$

⑦

Step 3 $\mathcal{F}_E$ 是单调类 (∵ $\mathcal{F}$ 是)

Step 4 如果 $E \in \mathcal{A}$，则 $\mathcal{A} \subset \mathcal{F}_E$

$\forall F \in \mathcal{A}$，$E \setminus F$，$F \setminus E$，$E \cap F \in \mathcal{A}$
$\subset \mathcal{F}$

Step 5 如果 $E \in \mathcal{A}$，则 $\mathcal{F} \subset \mathcal{F}_E$

由 Step 4. $\mathcal{A} \subset \mathcal{F}_E$

再由 Step 3 $\mathcal{F} \subset \mathcal{F}_E$

Step 6 如果 $E \in \mathcal{F}$，则 $\mathcal{F} \subset \mathcal{F}_E$

$$E \in \mathcal{F} \xRightarrow{\text{Step 5}} E \in \mathcal{F}_A, \ \forall A \in \mathcal{A}$$

$$\xRightarrow{\text{Step 2}} A \in \mathcal{F}_E, \ \forall A \in \mathcal{A}$$

$$\Longrightarrow \mathcal{A} \subset \mathcal{F}_E$$

$$\xRightarrow{\text{Step 3}} \mathcal{F} \subset \mathcal{F}_E$$

Step 7 $\mathcal{F}$ 是代数.

$\forall E, F \in \mathcal{F}$

Step 6

$\Longrightarrow E \in \mathcal{F} \subset \mathcal{F}_F$

$\Rightarrow E \setminus F,\ E \cap F \in \mathcal{F}$

而且 $X \in \mathcal{A} \subset \mathcal{F}$

$\Longrightarrow \mathcal{F}$ 对差补和有限并封闭.

Step 8 $\mathcal{F}$ 是 σ-代数.

设 $A_k \in \mathcal{F}$, $k = 1, 2, \cdots$

令 $A \stackrel{def}{=} \bigcup_{k=1}^{\infty} A_k$

$B_n \stackrel{def}{=} \bigcup_{k=1}^{n} A_k$

$\xRightarrow{\text{Step 7}} B_n \in \mathcal{F}$

$\xRightarrow{B_n \nearrow A} A \in \mathcal{F}$

$\mathcal{F}$ 是单调类

为证明 Tonelli，只需证：

Thm

设 $(X_1, \mathcal{M}_1, \mu_1)$
$(X_2, \mathcal{M}_2, \mu_2)$ σ-有限

$E \in \mathcal{M}_1 \otimes \mathcal{M}_2$

则

(i) $x \mapsto \mu_2(E_x)$ $\mathcal{M}_1$-可测

$y \mapsto \mu_1(E^y)$ $\mathcal{M}_2$-可测

(ii) $$(\mu_1 \times \mu_2)(E) = \int_{X_1} \mu_2(E^x)\, d\mu_1(x)$$

$$= \int_{X_2} \mu_1(E^y)\, d\mu_2(y)$$

Rmk: Thm $\Longleftrightarrow$ Tonelli holds for χ_E

$\Longrightarrow$ - - - - for 非负简单函数 on $X_1 \times X_2$

$\xRightarrow{\text{MCT}}$ Tonelli holds for $\forall f \in L^+(X_1 \times X_2)$

Pf of Thm

只證明 μ_1, μ_2 都是有限測度的情形

令

$$\mathcal{F} \overset{def}{=} \{ E \in \mathcal{M}_1 \otimes \mathcal{M}_2 : E \text{ 滿足 (i), (ii)} \}$$

Claim $\mathcal{F} = \mathcal{M}_1 \otimes \mathcal{M}_2$

Step 1. $\mathcal{M}_1 \times \mathcal{M}_2 \subset \mathcal{F}$

設 $E = A \times B$ with $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$

$$\Rightarrow \quad E_x = \begin{cases} B, & \text{if } x \in A \\ \phi & \text{if } x \notin A \end{cases}$$

$$E^y = \begin{cases} A & \text{if } y \in B \\ \phi & \text{if } y \notin B \end{cases}$$

$$\Rightarrow \begin{cases} \mu_2(E_x) = \mu_2(B)\chi_A(x) \\ \mu_1(E^y) = \mu_1(A)\chi_B(y) \end{cases}$$

$\Rightarrow$ (i) holds

且 $$\int_{X_1} \mu_2(E_x)\,d\mu_1(x) = \mu_2(B)\int_{X_1} \chi_A\,d\mu_1$$

$$= \mu_2(B)\mu_1(A) = (\mu_1 \times \mu_2)(E)$$

同理
$$(\mu_1\times\mu_2)(E) = \int_{X_2} \mu_1(E^y)\, d\mu_2(y)$$

Step 2 令

$\mathcal{A} \overset{\text{def}}{=} \{\mathcal{M}_1\times\mathcal{M}_2$ 中成员的有限不交并$\}$

$\overset{\text{Step 1}}{\Longrightarrow} \mathcal{A} \subset \mathcal{F}$

且 $\mathcal{A}$ 是 $X_1\times X_2$ 上的代数.

($\because$ $\mathcal{M}_1\times\mathcal{M}_2$ 是半代数).

Step 3 $\mathcal{F}$ 是单调类.

设 $\mathcal{F} \ni E_k \nearrow E$

令 $f_k(y) \overset{\text{def}}{=} \mu_1((E_k)^y)$, $y\in X_2$

$f(y) \overset{\text{def}}{=} \mu_1(E^y)$, $y\in X_2$

$E_k\in\mathcal{F} \Longrightarrow f_k$ $\mathcal{M}_2$-可测

$E_k\nearrow E \Longrightarrow (E_k)^y \nearrow E^y$

$$\Longrightarrow f(y) = \mu_1(E^y) = \lim_{k\to\infty}\mu_1((E_k)^y)$$

$\Rightarrow$ f $\mathcal{M}_2$-可测 且 $f_k \nearrow f$

$$\int_{X_2} \mu_1(E^y)\, d\mu_2(y)$$

$$= \int_{X_2} f(y)\, d\mu_2(y)$$

$$\overset{\text{MCT}}{=} \lim_{k\to\infty} \int_{X_2} \mu_1((E_k)^y)\, d\mu_2(y)$$

$$\overset{E_k \in \mathcal{F}}{=} \lim_{k\to\infty} (\mu_1 \times \mu_2)(E_k)$$

$$= (\mu_1 \times \mu_2)(E)$$

$\Rightarrow$ $E \in \mathcal{F}$

类似地，$\mathcal{F} \ni E_k \searrow E \Rightarrow E \in \mathcal{F}$

（这里用到 μ_1, μ_2 是有限测度）

Step 4 由 MCL，$\mathcal{M}_1 \otimes \mathcal{M}_2 \subset \mathcal{F}$