第二十八讲 (2026.6.8)

Thm 设 $\mathcal{A}$ 是 X 上的代数

μ_0 是 $\mathcal{A}$ 上的预测度.

$$\mu^*(E) \overset{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} \mu_0(A_k) : \begin{array}{l} A_k \in \mathcal{A},\ k=1,2,\cdots \text{ with} \\ E \subset \bigcup_{k=1}^{\infty} A_k \end{array} \right\}$$

(i) μ^* 是 X 上的外测度.

(ii) $\mu^*|_{\mathcal{A}} = \mu_0$

(iii) $\mathcal{A} \subset \mathcal{M} \overset{\text{def}}{=} \{\mu^*\text{-可测集}\}$

(iv) $\mu \overset{\text{def}}{=} \mu^*|_{\mathcal{M}}$ 是测度.

如果 ν 是 $\mathcal{M}$ 上的一个测度 with

$\nu|_{\mathcal{A}} = \mu_0$, 则

$$\nu(E) \le \mu(E), \quad \forall E \in \mathcal{M}$$

进而, 如果 $\mu(E) < +\infty$, 则

$$\nu(E) = \mu(E)$$

特别地, 如果 μ_0 是 σ-有限的, 则 $\nu = \mu$.

此时，μ 是 μ_0 从 $\mathcal{A}$ 到 $\mathfrak{M}$ 的唯一延拓 (2

称. μ_0 是 σ-有限的 是指 $\exists \{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$

s.t.
$$\begin{cases} \mu_0(A_k) < \infty, & k = 1, 2 \cdots \\ X = \bigcup_{k=1}^{\infty} A_k \end{cases}$$

Pf (iii) 设 $A \in \mathcal{A}$

Claim $A \in \mathfrak{M}$

i.e.
$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \forall E \subset X.$$

$\forall \varepsilon > 0$, $\exists A_k \in \mathcal{A}$, $k = 1, 2 \cdots$ s.t.
$$\begin{cases} E \subset \bigcup_{k=1}^{\infty} A_k \\ \sum_{k=1}^{\infty} \mu_0(A_k) < \mu^*(E) + \varepsilon \end{cases}$$

$$\Longrightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$
$$\leq \mu^*\left(\bigcup_{k=1}^{\infty} (A_k \cap A)\right) + \mu^*\left(\bigcup_{k=1}^{\infty} (A_k \cap A^c)\right)$$

次可加
$$\leq \sum_{k=1}^{\infty}\left[\mu^*(\underbrace{A_k \cap A}_{\in \mathcal{A}}) + \mu^*(\underbrace{A_k \cap A^c}_{\in \mathcal{A}})\right] \quad ③$$

$\mu^*|_{\mathcal{A}} = \mu_0$
$$= \sum_{k=1}^{\infty}\left[\mu_0(A_k \cap A) + \mu_0(A_k \cap A^c)\right]$$

μ_0 有限可加
$$= \sum_{k=1}^{\infty} \mu_0(A_k)$$

$$< \mu^*(E) + \varepsilon$$

$$\Longrightarrow \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

(iv) 设 $E \in \mathfrak{M}$

$\forall \{A_k\}_{k=1}^{\infty}$ with $E \subset \bigcup_{k=1}^{\infty} A_k$

$$\nu(E) \leq \sum_{k=1}^{\infty} \nu(A_k) = \sum_{k=1}^{\infty} \mu_0(A_k) \quad (\because \nu|_{\mathcal{A}} = \mu_0)$$

$$\Longrightarrow \nu(E) \leq \mu^*(E) = \mu(E)$$

当 $\mu(E) < \infty$ 时 要证明反向不等式

$\forall \varepsilon > 0.$ $\exists \{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ s.t.

$$\begin{cases} E \subset \bigcup_{k=1}^{\infty} A_k \\ \sum_{k=1}^{\infty} \mu_0(A_k) < \mu(E) + \varepsilon \end{cases}$$

$$A \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} A_k$$

$$\Rightarrow \quad \mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k) = \sum_{k=1}^{\infty} \mu_0(A_k)$$

$$< \mu(E) + \varepsilon$$

$$\Rightarrow \quad \mu(A \setminus E) = \mu(A) - \mu(E) < \varepsilon$$

$$\Rightarrow \quad \nu(A) = \lim_{N \to \infty} \nu\left(\bigcup_{k=1}^{N} A_k\right) \quad \left(\begin{array}{c}\text{[illegible]下の}\\ \text{連続性}\end{array}\right)$$

$$= \lim_{N \to \infty} \mu_0\left(\bigcup_{k=1}^{N} A_k\right) \quad (\nu|_{\mathcal{A}} = \mu_0)$$

$$= \lim_{N \to \infty} \mu\left(\bigcup_{k=1}^{N} A_k\right) \quad (\mu|_{\mathcal{A}} = \mu_0)$$

$$= \mu(A)$$

$$\Rightarrow \quad \mu(E) \leq \mu(A) = \nu(A)$$

$$= \nu(E) + \nu(A \setminus E)$$

$$\leq \nu(E) + \mu(A \setminus E) \quad (\because \nu \leq \mu)$$

$$\leq \nu(E) + \varepsilon$$

$$\Rightarrow \quad \mu(E) \leq \nu(E)$$

如果 μ_0 是 σ-有限的

$\Rightarrow \exists \{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ s.t.

$$\begin{cases} \mu_0(A_k) < \infty, & \forall k \\ X = \bigsqcup_{k=1}^{\infty} A_k \end{cases}$$

$\forall E \in \mathcal{M}$

$$\nu(E) = \sum_{k=1}^{\infty} \nu(E \cap A_k)$$

$$= \sum_{k=1}^{\infty} \mu(E \cap A_k) = \mu(E).$$

Def 如果 $\mathcal{F} \subset 2^X$ s.t.

(i) $\phi \in \mathcal{F}$

(ii) 对有限交封闭, i.e.

$E_1, E_2 \in \mathcal{F} \Rightarrow E_1 \cap E_2 \in \mathcal{F}$

(iii) $E \in \mathcal{F} \Rightarrow E^c$ 可表为 $\mathcal{F}$ 中成员的有限不交并.

则称 $\mathcal{F}$ 是 X 上一个半代数.

(elementary family)

例：{$\mathbb{R}$中的区间} 是 $\mathbb{R}$ 上半代数 ⑥

Prop 1 $\mathcal{F}$ — X 上的半代数.

则 $\mathcal{A} \overset{\text{def}}{=}$ {$\mathcal{F}$ 中成员的有限不交并}

是 X 上的代数.

Pf 设 $A, B \in \mathcal{F}$

$\Rightarrow \quad B^c = \bigsqcup_{k=1}^{n} C_k \quad$ for some $C_k \in \mathcal{F}$

$\Rightarrow \quad A \setminus B = \bigsqcup_{k=1}^{n} \underbrace{(A \cap C_k)}_{\in \mathcal{F}}$

$\Rightarrow \quad A \cup B = (A \setminus B) \sqcup B$

$= \bigsqcup_{k=1}^{n} (A \cap C_k) \sqcup B \in \mathcal{A}$

$\Rightarrow$ $\mathcal{A}$ 对有限并封闭

$\forall E \in \mathcal{A}$

$\Rightarrow \quad E = \bigsqcup_{k=1}^{n} A_k \quad$ with $A_k \in \mathcal{F}$

$$A_k^c = \bigsqcup_{j=1}^{m_k} C_j^{(k)} \quad \text{with} \quad C_j^{(k)} \in \mathcal{F}$$ ⑦

（$\mathcal{F}$ 是半代数）

$$\Rightarrow \quad E^c = \bigcap_{k=1}^{n} A_k^c$$

$$= \bigcap_{k=1}^{n} \bigsqcup_{j=1}^{m_k} C_j^{(k)}$$

$$= \bigcup_{\substack{1 \le j_k \le m_k \\ 1 \le k \le n}} C_{j_1}^{(1)} \cap \cdots \cap C_{j_n}^{(n)} \in \mathcal{A}$$

乘积测度空间

$(X_1, \mathcal{M}_1, \mu_1)$

$(X_2, \mathcal{M}_2, \mu_2)$ $\overset{?}{\Rightarrow}$ $(X_1 \times X_2, \mathcal{M}_1 \times \mathcal{M}_2, \mu_1 \times \mu_2)$

Def $\mathcal{M}_1 \otimes \mathcal{M}_2 \overset{def}{=} \mathcal{M}_1 \times \mathcal{M}_2$ 生成的 σ-代数

i.e. 包含 $\mathcal{M}_1 \times \mathcal{M}_2$ 的最小的 σ-代数

可测矩形

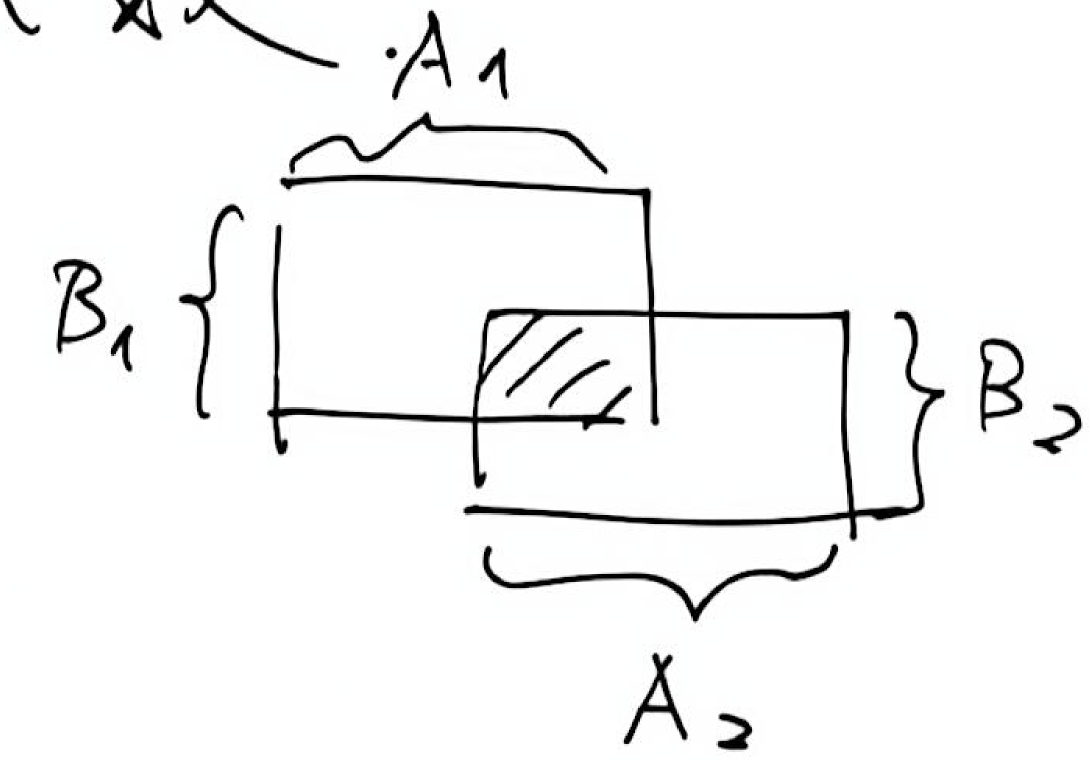
$$\mathcal{M}_1 \times \mathcal{M}_2 \overset{def}{=} \{ A \times B : A \in \mathcal{M}_1, B \in \mathcal{M}_2 \}$$

Q: 如何定义 $\mu_1 \times \mu_2$ s.t. Fubini?

首先要满足

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A)\,\mu_2(B)$$

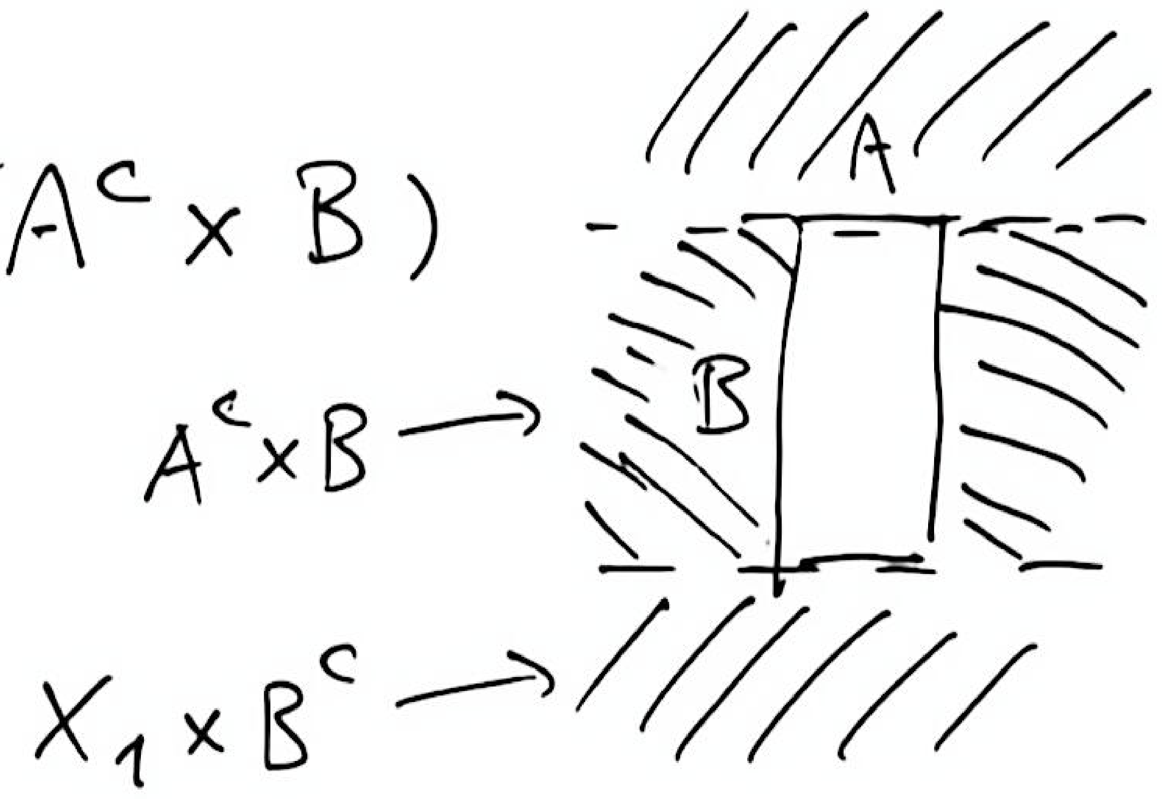
Prop 2 $\mathcal{M}_1 \times \mathcal{M}_2$ 是半代数



Pf $(A_1 \times B_1) \cap (A_2 \times B_2)$

$$= (A_1 \cap A_2) \times (B_1 \cap B_2)$$

$$(A \times B)^c = (X_1 \times B^c) \sqcup (A^c \times B)$$



Def

$\mathcal{A} \overset{\text{def}}{=} \{$可测矩形的有限不交并$\}$

是 $X_1 \times X_2$ 上的代数 (by prop 1).

令 $\mu_0\left(\bigsqcup_{k=1}^{n} (A_k \times B_k)\right) \overset{\text{def}}{=} \sum_{k=1}^{n} \mu_1(A_k)\,\mu_2(B_k)$

$\Rightarrow$ μ_0 是 $\mathcal{A}$ 上的预测度.

对 $E \subset X_1 \times X_2$

$$\mu^*(E) \overset{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} \mu_0(E_k) : \begin{array}{l} E_k \in \mathcal{A},\ k=1,2\cdots \\ E \subset \bigcup_{k=1}^{\infty} E_k \end{array} \right\}$$

$\Rightarrow$ μ^* 是 $X_1 \times X_2$ 上的外测度.

$\mathcal{M} \overset{\text{def}}{=} \{\mu^*\text{-可测集}\}$

$\mu \overset{\text{def}}{=} \mu^*|_{\mathcal{M}}$ 是完备测度.

$\mathcal{M}_1 \otimes \mathcal{M}_2 \subset \mathcal{M}$.

$(\because\ \mathcal{M}_1 \times \mathcal{M}_2 \subset \mathcal{A} \subset \mathcal{M})$.

Def $\mu_1 \times \mu_2 \overset{\text{def}}{=} \mu^*|_{\mathcal{M}_1 \otimes \mathcal{M}_2}$

Rmk：一般来说，$\mu_1 \times \mu_2$ 不完备

除非 $\mathcal{M}_1 \otimes \mathcal{M}_2 = \mathcal{M}$.