第二十四讲

上次课.

Thm: 单调函数 f a.e. 可微 且 $f' \in L^1[a,b]$

Cor BV函数 — — — — —

C-L 函数 s.t.

(i) $f \nearrow$ on $[0,1]$

(ii) $f \in C[0,1]$

(iii) $f' = 0$ a.e.

Def 设 $f:[a,b] \to \mathbb{R}$

如果 $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. 对 $[a,b]$ 的任意有限个互不相交的开子区间 $\{(\alpha_k, \beta_k)\}_{k=1}^N$

只要
$$\sum_{k=1}^N (\beta_k - \alpha_k) < \delta$$
就有
$$\sum_{k=1}^N |f(\beta_k) - f(\alpha_k)| < \varepsilon$$
则称 f 在 $[a,b]$ 上绝对连续.

$AC[a,b] \overset{\text{def}}{=} \{[a,b]$ 上绝对连续函数$\}$ ②

例: C-L 函数 $\notin AC[0,1]$

例: $\{$Lipschitz 函数$\} \subset AC[a,b]$

$\exists L > 0$ s.t.

$$|f(x') - f(x'')| \leq L|x' - x''|, \quad \forall x', x'' \in [a,b]$$

(HW: Ex. 32)

<u>Prop 1</u> $AC[a,b]$ 是向量空间.

<u>Prop 2</u> $AC[a,b] \subset BV[a,b]$

Pf 设 $f \in AC[a,b]$

$\Rightarrow$ (取 $\varepsilon = 1$), $\exists \delta > 0$ s.t.

$\forall (\alpha_k, \beta_k) \subset [a,b]$, $k = 1, 2, \cdots, N$ 互不相交

with $$\sum_{k=1}^{N} (\beta_k - \alpha_k) < \delta$$

均有

$$\sum_{k=1}^{N} |f(\beta_k) - f(\alpha_k)| \leq 1. \qquad (*)$$

③

令 $J \stackrel{\text{def}}{=} \left[\frac{b-a}{\delta}\right] + 1$

$t_j \stackrel{\text{def}}{=} a + \frac{b-a}{J} j$, $j = 0, 1, 2, \cdots, J$.

$\Rightarrow P = \{t_0, t_1, \cdots, t_J\}$ 是 $[a, b]$ 的划分

s.t. $V_{t_{j-1}}^{t_j}(f) \leq 1$ (by $(*)$)

$\Rightarrow V_a^b(f) = \sum_{j=1}^{J} V_{t_{j-1}}^{t_j}(f) \leq J$.

Cor $f \in AC[a,b] \Rightarrow \begin{cases} f \text{ a.e. 可微} \\ f' \in L^1[a,b] \end{cases}$

Prop 3 $f \in AC[a,b] \Rightarrow f$ 把零集映为零集

(HW Ex. 19)

Prop 4 $\left.\begin{array}{l} f \in L^1[a,b] \\ F(x) = \int_a^x f(t)\,dt \end{array}\right\} \Rightarrow F \in AC[a,b]$

Pf $f \in L^1[a,b]$

$\Rightarrow \forall \varepsilon > 0$. $\exists \delta > 0$ s.t.

$$\int_E |f|\,dm < \varepsilon,$$

$$\forall E \subset [a,b] \text{ with } m(E) < \delta.$$

(積分の絶対連続性)

$\forall\ (\alpha_k, \beta_k) \subset [a, b]$, $k = 1, 2, \cdots, N$ 互不相交

with

$$\sum_{k=1}^{N} (\beta_k - \alpha_k) < \delta$$

$$\Longrightarrow \sum_{k=1}^{N} |F(\beta_k) - F(\alpha_k)|$$

$$\leq \sum_{k=1}^{N} \int_{\alpha_k}^{\beta_k} |f(t)|\,dt$$

$$= \int_{\bigsqcup_{k=1}^{N} (\alpha_k, \beta_k)} |f|\,dm < \varepsilon$$

<u>Prop</u> 5 $\quad f \in AC[a, b]$

$$\Longrightarrow x \longmapsto V_a^x(f) \in AC[a, b]$$

<u>(1)</u>

$$V_a^x(f) = \int_a^x |f'(t)|\,dt$$

(<u>HW</u>. Ex. 1.6)

Thm
$$\left.\begin{array}{l} f \in AC[a,b] \\ f' = 0 \quad a.e. \end{array}\right\} \Rightarrow f = \text{const}.$$

Pf 假设 $f \not\equiv \text{const}$.

$\Rightarrow$ $\exists\, c \in (a, b]$ s.t.

$$f(c) \neq f(a)$$

令

$$\varepsilon_0 \overset{\text{def}}{=} \frac{1}{3} |f(c) - f(a)|$$

$f \in AC[a,b]$

$\Rightarrow$ $\exists\, \delta_0 > 0$ s.t. $\forall\, (\alpha_j, \beta_j) \subset (a, c)$

$j = 1, 2, \cdots, n$ 互不相交, with

$$\sum_{j=1}^{n} (\beta_j - \alpha_j) < \delta_0$$

均有

$$\sum_{j=1}^{n} |f(\beta_j) - f(\alpha_j)| < \varepsilon_0$$

令

$$E \overset{\text{def}}{=} \{ x \in (a, c) : f'(x) = 0 \}.$$

$$\eta \overset{\text{def}}{=} \frac{\varepsilon_0}{2(b-a)}$$

$\Rightarrow \quad \forall x \in E, \quad \exists\, h_x^{(k)} \to 0^+$ with

$[x, x+h_x^{(k)}] \subset (a, c)$, s.t.

$$|f(x+h_x^{(k)}) - f(x)| < \eta\, |h_x^{(k)}|$$

$k = 1, 2, \ldots$

$$\Rightarrow \Gamma \overset{\text{def}}{=} \left\{ [x, x+h_x^{(k)}] \right\}_{x \in E,\, k \in \mathbb{N}}$$

是 E 的 Vitali 覆盖.

$\overset{\text{Vitali}}{\Longrightarrow} \quad \exists\, [x_1, x_1+h_1], \ldots, [x_N, x_N+h_N] \in \Gamma$

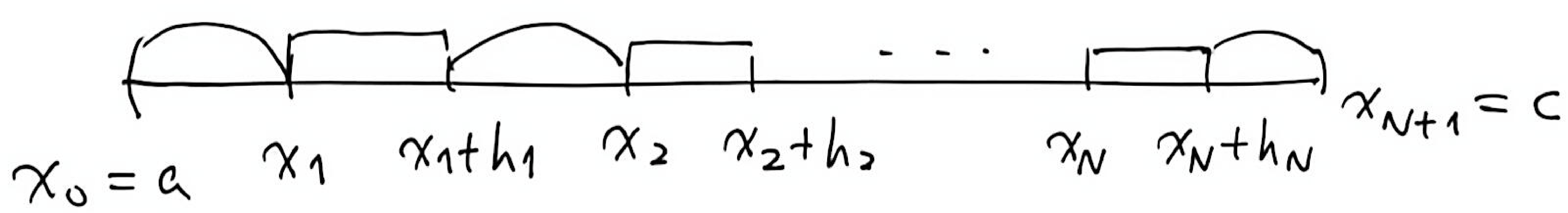
互不相交 s.t.

$$m\left((a,c) \setminus \bigsqcup_{k=1}^{N} [x_k, x_k+h_k]\right)$$

$$= m\left(E \setminus \bigsqcup_{k=1}^{N} [x_k, x_k+h_k]\right) < \delta_0$$

不妨设这些区间依次排列

$$a = x_0 < x_1 < x_1+h_1 < x_2 < \cdots < x_N + h_N < x_{N+1} = c$$



$$(a, c) \setminus \bigsqcup_{k=1}^{N} [x_k, x_k + h_k]$$ ⑦

$$= (x_0, x_1) \sqcup (x_1 + h_1, x_2) \sqcup \cdots \sqcup (x_N + h_N, x_{N+1})$$

$$\Longrightarrow \sum_{k=0}^{N} [x_{k+1} - (x_k + h_k)] < \delta_0$$

$$(\leftarrow h_0 = 0)$$

$$\overset{f \in AC}{\Longrightarrow} \sum_{k=0}^{N} |f(x_{k+1}) - f(x_k + h_k)| < \varepsilon_0$$

$$\Longrightarrow 3\varepsilon_0 = |f(c) - f(a)|$$

$$\leq \underbrace{\sum_{k=0}^{N} |f(x_{k+1}) - f(x_k + h_k)|}_{< \varepsilon_0} + \underbrace{\sum_{k=0}^{N} |f(x_k + h_k) - f(x_k)|}_{< \eta \sum_{k=1}^{N} h_k}$$

$$< \varepsilon_0 + (b-a)\eta$$

$$< 2\varepsilon_0$$

矛盾.

Thm 2

(i) $F \in AC[a,b] \Longrightarrow \begin{cases} F \text{ a.e. 可微} \\ F' \in L^1[a,b] \end{cases}$

且 $$F(x) - F(a) = \int_a^x F'(t)\,dt \qquad x \in [a,b]$$

(ii) $f \in L^1[a,b] \Longrightarrow \exists F \in AC[a,b]$ s.t. $F' = f$ a.e.

(事实上，$F(x) \overset{def}{=} \int_a^x f(t)\,dt$ 即可).

Rmk: Newton-Leibniz $\Longleftrightarrow F \in AC[a,b]$.

Pf (i) 前一结论见 Cor.

令 $$G(x) \overset{def}{=} \int_a^x F'(t)\,dt, \qquad x \in [a,b]$$

LDT $\Longrightarrow G' = F'$ a.e.

由 Prop 4. $G \in AC[a,b]$

$\Longrightarrow F - G \in AC[a,b]$ s.t. $(F-G)' = 0$ a.e.

Thm 1

$\Longrightarrow \quad F - G = \text{const}.$

令 $\quad C \overset{\text{def}}{=} F(x) - \int_a^x F'(t)\,dt$

取 $x = a$

$\Longrightarrow \quad C = F(a)$

$\Longrightarrow \quad \int_a^x F'(t)\,dt = F(x) - F(a)$

(ii) By Prop 4 + LDT.

HW: Ex. 20.