第二十讲 (2026. 5. 11)

Thm (LDT)

$$f \in L^1_{loc} \Rightarrow \lim_{r\to 0^+} \frac{1}{m(B_r(x))}\int_{B_r(x)} f\,dm = f(x)$$

for a.e. $x \in \mathbb{R}^n$

Def 设 $E \in \mathcal{L}$, $x \in \mathbb{R}^n$, 如果

$$\lim_{r\to 0^+} \frac{m(E\cap B_r(x))}{m(B_r(x))} = 1$$

则称 x 是 E 的一个密度点

Thm (Lebesgue 密度定理)

(i) a. e. $x \in E$ 都是 E 的密度点

(ii) 对 a. e. $x \in E^c$,

$$\lim_{r\to 0^+} \frac{m(E\cap B(x,r))}{m(B(x,r))} = 0$$

Pf $\dfrac{m(E\cap B_r(x))}{m(B_r(x))} = \dfrac{1}{m(B_r(x))}\displaystyle\int_{B_r(x)} \chi_E\,dm$

$\to \chi_E(x)$ as $r \to 0^+$

for a.e. $x \in \mathbb{R}^n$

②

Def 设 $f \in L^1_{loc}$，$x \in \mathbb{R}^n$. 如果

$$\lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0$$

则称 x 是 f 的 Lebesgue 点.

Thm $f \in L^1_{loc} \Rightarrow$ a.e. $x \in \mathbb{R}^n$ 都是 f 的 Lebesgue 点.

Pf 令 $L_f \stackrel{\text{def}}{=} f$ 的 Lebesgue 点全体.

要证明：

$$m(\mathbb{R}^n \setminus L_f) = 0$$

$\forall q \in \mathbb{Q}$，$\exists E_q \in \mathcal{L}$ with $m(E_q) = 0$，s.t.

$$\lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - q| \, dy = |f(x) - q|$$

$$\forall x \in \mathbb{R}^n \setminus E_q$$

（对 $|f-q|$ 应用 LDT）

令 $E \stackrel{\text{def}}{=} \bigcup_{q \in \mathbb{Q}} E_q$

$\Rightarrow m(E) = 0$

Claim $\mathbb{R}^n \setminus E \subset L_f$

$\forall x \in \mathbb{R}^n \setminus E = \bigcap_{q \in \mathbb{Q}} (\mathbb{R}^n \setminus E_q)$, 不妨设

$f(x)$ 有限 ($\because f$ a.e. 有限)

$\forall \varepsilon > 0$, $\exists q \in \mathbb{Q}$ s.t.

$$|f(x) - q| < \varepsilon/2$$

$$\Rightarrow \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy$$

$$\leq \underbrace{\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - q|\, dy}_{\to |f(x) - q| \text{ as } r \to 0^+} + |q - f(x)|$$

($\because x \in \mathbb{R}^n \setminus E_q$)

$$\Rightarrow \limsup_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy$$

$$\leq 2|f(x) - q| < \varepsilon$$

$$\Rightarrow \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0.$$

Def 设 $x \in \mathbb{R}^n$. 如果 $\mathcal{F}_x \subset \mathcal{L}$ 满足

(i) $\forall \varepsilon > 0$, $\exists E \in \mathcal{F}_x$ s.t. $\operatorname{diam} E < \varepsilon$

(ii) $\exists c > 0$ s.t.

$$m(E) > c\, m(B^E(x)), \quad \forall E \in \mathcal{F}_x$$

其中 $B^E(x)$ 是以 x 为中心, 包含 E 的最小开球

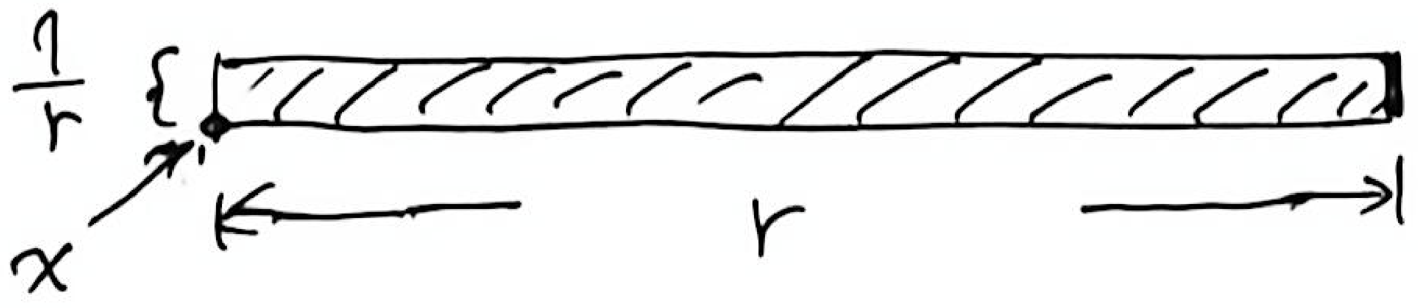
则称 $\mathcal{F}_x$ 正则收缩于 x.

例: $\{$包含 x 的开球$\}$

$\{$包含 x 的方体$\}$ 都正则收缩于 x

$\{B_{2r}(x) \setminus B_r(x)\}_{r>0}$

例: $\{$包含 x 的长方体$\}$ 不正则收缩于 x



R_r

$$B^{R_r}(x) \approx B_r(x)$$

$$m(R_r) = 1$$

$$m(B^{R_r}(x)) \approx v_n \cdot r^n \quad \text{with } v_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

Cor 设 $f \in L^1_{loc}$，$x \in L_f$

$\mathcal{F}_x \subset \mathcal{L}$ 正则收缩于 x，则

$$\lim_{\substack{\operatorname{diam} E \to 0 \\ E \in \mathcal{F}_x}} \frac{1}{m(E)} \int_E f \, dm = f(x).$$

Pf
$$\frac{1}{m(E)} \int_E |f(y) - f(x)| \, dy$$

$$\leq \frac{1}{c} \frac{1}{m(B^E(x))} \int_{B^E(x)} |f(y) - f(x)| \, dy$$

$$\to 0 \quad \text{as} \quad \operatorname{diam} B^E(x) \to 0$$

Cor $f \in L^1_{loc} \Rightarrow \displaystyle\lim_{\substack{m(B) \to 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f \, dm = f(x)$

for a.e. $x \in \mathbb{R}^n$

Cor $f \in L^1_{loc}(\mathbb{R}^1) \Rightarrow$ 对 a.e. $x \in \mathbb{R}^1$

$$\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(y) \, dy = f(x)$$

$$\lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^{x} f(y) \, dy = f(x)$$

$\Rightarrow$ $F' = f$ a.e.

where $F(x) = \displaystyle\int_a^x f(t) \, dt$

例 $\varphi \overset{\text{def}}{=} \frac{1}{v_n} \chi_{B_1(0)}$

$\Rightarrow \int \varphi \, dm = 1$

令 $\varphi_t(x) \overset{\text{def}}{=} t^{-n} \varphi(t^{-1}x), \quad x \in \mathbb{R}^n$

$$\Rightarrow (f * \varphi_t)(x) = \frac{1}{v_n t^n} \int_{\mathbb{R}^n} f(x-y) \chi_{B_1(0)}(t^{-1}y)\, dy$$

$$= \frac{1}{v_n t^n} \int_{B_t(0)} f(x-y)\, dy$$

$$= \frac{1}{m(B_t(x))} \int_{B_t(x)} f(y)\, dy$$

$\to f(x)$ as $t \to 0^+$

for a.e. $x \in \mathbb{R}^n$

(by LDT)

Def 設 $\{K_t\}_{t>0}$ 為一族可測函數 s.t.

(A1) $\int K_t \, dm = 1$

(A2) $\exists C_1 > 0$ s.t.

$$|K_t(x)| \le \frac{C_1}{t^n}, \quad \forall t \in (0,1)$$

(A3) $\exists\, C_2 > 0$ s.t.

$$|K_t(x)| \le \frac{C_2\, t}{|x|^{n+1}}, \quad \forall t>0,\ \forall x \in \mathbb{R}^n \setminus \{0\}$$

则称 $\{K_t\}_{t>0}$ 是一个恒等元逼近

(A.I. = Approximations to the Identity)

Thm 设 $\{K_t\}_{t>0}$ 是 A.I.，则：$\forall f \in L^1$

$$f * K_t \to f \quad \text{a.e.} \quad \text{as } t \to 0^+$$

Lem 设 $f \in L^1$. $x \in L_f$

$$g(r) \overset{\text{def}}{=} \frac{1}{r^n} \int_{|y| \le r} |f(x-y) - f(x)|\, dy$$

则 (i) $g \in C(0, +\infty)$

(ii) $\lim\limits_{r \to 0^+} g(r) = 0$

(iii) g 有界

Pf (i) $\forall r \in (0, +\infty)$

$$g(r+h) - g(r)$$

$$= \frac{1}{(r+h)^n} \int_{B_{r+h}(0) \setminus B_r(0)} |f(x-y) - f(x)|\, dy$$

$$+ \left[\frac{1}{(r+h)^n} - \frac{1}{r^n}\right] \int_{B_r(0)} |f(x-y) - f(x)|\, dy$$

$= \mathrm{I} + \mathrm{II}$

$\mathrm{I} \to 0$ as $h \to 0$ (by 积分的绝对连续性)

$\mathrm{II} \to 0$ as $h \to 0$ (由 $\frac{1}{r^n}$ 的连续性)

(ii) $\lim\limits_{r\to 0+} g(r) = v_n \lim\limits_{r\to 0+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy$

$= 0 \quad (\because x \in L_f)$

(iii) $\left.\begin{array}{l} g \in C(0,1] \\ g(0+) \text{ 存在} \end{array}\right\} \Rightarrow g$ 在 $(0,1]$ 上有界

而当 $r > 1$ 时

$$g(r) \leq \frac{1}{r^n} \int_{|y|\leq r} |f(x-y)| dy + v_n |f(x)|$$

$$\leq \|f\|_1 + v_n |f(x)|$$

Pf of Thm

$$|(f * K_t)(x) - f(x)|$$

$$\leq \int |f(x-y) - f(x)| |K_t(y)| dy$$

$$= \int_{|y|\leq t} + \sum_{k=0}^{\infty} \int_{2^k t < |y| \leq 2^{k+1} t}$$

$$\int_{|y| \le t} |f(x-y) - f(x)| \, |K_t(y)| \, dy$$

$$\le \frac{C_1}{t^n} \int_{|y| \le t} |f(x-y) - f(x)| \, dy \qquad (\text{by } (A2))$$

$$= C_1 \, g(t)$$

$$\int_{2^k t < |y| \le 2^{k+1} t} |f(x-y) - f(x)| \, |K_t(y)| \, dy$$

$$\overset{(A3)}{\le} \frac{C_2 \, t}{(2^k t)^{n+1}} \int_{|y| \le 2^{k+1}} |f(x-y) - f(x)| \, dy$$

$$= \frac{C_2 \cdot 2^n}{2^k} \, \frac{1}{(2^{k+1} t)^n} \int_{|y| \le 2^{k+1} t} |f(x-y) - f(x)| \, dy$$

$$= \frac{C_3}{2^k} \, g(2^{k+1} t)$$

$$\Rightarrow \; |(f * K_t)(x) - f(x)|$$

$$\le C \left[g(t) + \sum_{k=0}^{\infty} \frac{1}{2^k} \, g(2^{k+1} t) \right]$$

$$M \overset{\text{def}}{=} \sup_{t \in (0, +\infty)} |g(t)|$$

$\forall \varepsilon > 0$, $\exists N$ s.t.

$$\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \varepsilon$$

而当 t 充分小时

$$\begin{cases} g(t) < \varepsilon \\ g(2^{k+1}t) < \dfrac{\varepsilon}{N}, \quad k = 0, 1, 2, \cdots, N-1 \end{cases}$$

$$\Rightarrow \quad |(f * K_t)(x) - f(x)|$$

$$\leq C\left[\varepsilon + N \cdot \frac{\varepsilon}{N} + M \cdot \varepsilon\right]$$

当 t 充分小时.

HW: Chpt 3. Ex. 4