第十九讲 (2026.5.9)

$$\left.\begin{array}{l} f \in L^1[a,b] \\ F(x) = \int_a^x f(t)\,dt \end{array}\right\} \overset{?}{\Longrightarrow} \left\{\begin{array}{l} F \quad \text{a.e. 可微} \\ F' = f \quad \text{a.e.} \end{array}\right.$$

$$F'(x) = f(x) \iff \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(y)\,dy = f(x)$$

Q: $\displaystyle \lim_{\substack{|I|\to 0 \\ I \ni x}} \frac{1}{|I|}\int_I f(y)\,dy = f(x)$ for a.e. x?

Q: $\displaystyle f \in L^1 \overset{?}{\Longrightarrow} \lim_{\substack{m(B)\to 0 \\ B \ni x}} \frac{1}{m(B)}\int_B f\,dm = f(x)$ for a.e. $x \in \mathbb{R}^n$

以下 B 总代表开球

Def

$$L^1_{loc} \overset{def}{=} \{ f \text{ 可测} : f \in L^1(K),\ \forall K \overset{cpt}{\subset} \mathbb{R}^n \}$$

$$= \{ \text{局部可积函数} \}$$

Thm (Lebesgue 微分定理, LDT)

$$f \in L^1_{loc} \Longrightarrow \lim_{r\to 0} \frac{1}{m(B_r(x))}\int_{B_r(x)} f\,dm = f(x)$$

for a.e. $x \in \mathbb{R}^n$

<u>Def</u> 对 $f \in L^1_{loc}$,

$$f^*(x) \overset{def}{=} \sup_{B \ni x} \frac{1}{m(B)} \int_B |f|\, dm, \qquad x \in \mathbb{R}^n$$

称为 f 的非中心 H-L 极大函数.

$$(Mf)(x) \overset{def}{=} \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|\, dm$$

称为 f 的中心 H-L 极大函数.

<u>HW</u>: 证明: $Mf(x) \leq f^*(x) \leq 2^n Mf(x)$

<u>Prop</u> $f \in L^1_{loc} \implies Mf$ 下半连续.

i.e. $\forall \alpha \in \mathbb{R}$

$$\{Mf > \alpha\} \overset{open}{\subset} \mathbb{R}^n$$

<u>Pf</u> $x \in \{Mf > \alpha\} \iff \alpha < Mf(x)$

$\implies \exists r > 0$ s.t.

$$\alpha < \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|\, dm$$

$\implies \exists \rho > r$ s.t.

$$\alpha < \frac{1}{m(B_\rho(x))} \int_{B_r(x)} |f|\, dm$$

$\forall y \in B_{\rho-r}(x)$.

$$B_r(x) \subset B_\rho(y)$$

$$\left[\begin{array}{l} \forall z \in B_r(x). \\ \qquad |z-y| \leq |z-x| + |x-y| < r + \rho - r = \rho \end{array} \right]$$

$$\Rightarrow \quad \alpha < \frac{1}{m(B_\rho(y))} \int_{B_r(x)} |f| \, dm$$

$$\leq \frac{1}{m(B_\rho(y))} \int_{B_\rho(y)} |f| \, dm \leq Mf(y)$$

$$\Rightarrow \quad B_{\rho-r}(x) \subset \{ Mf > \alpha \}$$

<u>Cor</u> $\quad f \in L^1_{loc} \Rightarrow Mf$ 可测

<u>HW</u>: $\quad f \in L^1_{loc} \Rightarrow f^*$ 可测

<u>Thm</u> (H-L 极大定理)

算子 $M: L^1 \to L^+$, $f \mapsto Mf$ 是弱(1, 1) 型的; i.e. $\exists C > 0$ s.t. $\forall \alpha > 0$

$$m(\{ Mf > \alpha \}) \leq \frac{C}{\alpha} \|f\|_1, \quad \forall f \in L^1$$

Cor $f \in L^1 \Rightarrow Mf$ a.e. 有限.

Pf $\{Mf = +\infty\} = \bigcap_{\alpha > 0} \{Mf > \alpha\}$

$\Rightarrow$ $m(\{Mf = +\infty\}) \leq m(\{Mf > \alpha\})$

$\leq \frac{1}{\alpha} \|f\|_1$

$\to 0$ as $\alpha \to +\infty$

Thm (Vitali 覆盖引理)

设 $\mathcal{B} = \{B_1, \cdots, B_N\}$ 是有限个开球

则 $\exists B_{k_1}, \cdots, B_{k_p} \in \mathcal{B}$ s.t.

$$\sum_{j=1}^{p} m(B_{k_j}) \geq \frac{1}{3^n} m\left(\bigcup_{k=1}^{N} B_k\right)$$

Pf 令

$B^* \overset{def}{=} B$ 的同心球, with

$\mathrm{diam}(B^*) = 3\, \mathrm{diam}(B)$

Note: $\forall B, B' \in \mathcal{B}$

$$\left.\begin{array}{l} B \cap B' \neq \phi \\ \mathrm{diam}\, B' \leq \mathrm{diam}\, B \end{array}\right\} \Rightarrow B' \subset B^*$$

⑤

从 $\mathcal{B}$ 中选出半径最大的球 B_{k_1}

（若不唯一则任选其一）并从 $\mathcal{B}$ 中删除

与 B_{k_1} 相交的球（这些球可被 $B_{k_1}^*$ 覆盖）

余下的球的全体记为 $\mathcal{B}_1$

对 $\mathcal{B}_1$ 做同样的操作，i.e. 选其中最大

的球 B_{k_2} 并删除与之相交者.

$\vdots$

$\Rightarrow$ 最后得到 $B_{k_1}, \cdots, B_{k_p}$. 互不相交.

（与之相交者已被删除）

<u>Claim</u> $\displaystyle \sum_{j=1}^{p} m(B_{k_j}) \geq \frac{1}{3^n} m\Big(\bigcup_{k=1}^{N} B_k\Big)$

$\forall B \in \mathcal{B}$, $\exists j \in \{1, \cdots, p\}$ s.t.

$$\begin{cases} B \cap B_{k_j} \neq \phi \\ \operatorname{diam} B \leq \operatorname{diam} B_{k_j} \end{cases}$$

（由 $B_{k_1}, \cdots, B_{k_p}$ 的取法，B 要么是 $B_{k_1}, \cdots, B_{k_p}$ 中的一个，要么是某一步被删除的）

$\Rightarrow B \subset B_{k_j}^*$

$$\Rightarrow \bigcup_{k=1}^{N} B_k \subset \bigcup_{j=1}^{p} B_{k_j}^*$$

$$\Rightarrow m\left(\bigcup_{k=1}^{N} B_k\right) \leq \sum_{j=1}^{p} m(B_{k_j}^*) = 3^n \sum_{j=1}^{p} m(B_{k_j})$$

Pf of H-L 极大定理

令

$$E_\alpha \overset{\text{def}}{=} \{ Mf > \alpha \}$$

$\forall x \in E_\alpha$, $\exists r_x > 0$ s.t.

$$\frac{1}{m(B_{r_x}(x))} \int_{B_{r_x}(x)} |f| \, dm > \alpha$$

$$\Rightarrow m(B_{r_x}(x)) < \frac{1}{\alpha} \int_{B_{r_x}(x)} |f| \, dm$$

$\forall K \overset{\text{cpt}}{\subset} E_\alpha$, $\exists B_1, \cdots, B_N \in \{ B_{r_x}(x) \}_{x \in E_\alpha}$

s.t.

$$K \subset \bigcup_{k=1}^{N} B_k$$

$\overset{\text{Vitali}}{\Longrightarrow} \exists B_{k_1}, \cdots, B_{k_p} \in \{ B_1, \cdots, B_N \}$

互不相交 s.t.

$$m\left(\bigcup_{k=1}^{N} B_k\right) \leq 3^n \sum_{j=1}^{p} m(B_{k_j})$$

$$\Rightarrow m(K) \leq 3^n \sum_{j=1}^{p} m(B_{k_j})$$

$$\leq 3^n \sum_{j=1}^{p} \frac{1}{\alpha} \int_{B_{k_j}} |f| \, dm$$

$$= \frac{3^n}{\alpha} \int_{\bigsqcup_{j=1}^{p} B_{k_j}} |f| \, dm$$

$$\leq \frac{3^n}{\alpha} \|f\|_1 .$$

$$\Rightarrow m(E_\alpha) = \sup \{ m(K) : K \overset{cpt}{\subset} E_\alpha \}$$

$$\leq \frac{3^n}{\alpha} \|f\|_1 .$$

Pf of LDT

Step 1 先假設 $f \in C(\mathbb{R}^n)$

$\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$. $\exists \delta > 0$ s.t.

$$|f(y) - f(x)| < \varepsilon , \quad \forall y \in B_\delta(x)$$

$\Rightarrow \forall r < \delta$

$$\left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) \, dy - f(x) \right|$$

$$\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy$$

$$< \varepsilon$$

<u>Step 2</u> 一般情形.

不妨设 $f \in L^1$ （否则代以 $f \cdot \chi_B$ ）

令

$$E \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, dm - f(x) \right| > 0 \right\}$$

$\Longrightarrow$ 只需证: $m(E) = 0$

令

$$E_\alpha \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, dm - f(x) \right| > 2\alpha \right\}$$

$$\Longrightarrow \quad E = \bigcup_{k=1}^{\infty} E_{\frac{1}{k}}$$

$\Longrightarrow$ 转化为:

<u>Claim</u> $\forall \alpha, \quad m(E_\alpha) = 0$

$\forall \varepsilon > 0, \ \exists g \in C_c(\mathbb{R}^n)$ s.t.

$$\|f - g\|_1 < \varepsilon \qquad (\because C_c(\mathbb{R}^n) \overset{\text{dense}}{\subset} L$$

$$\left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy - f(x) \right|$$

$$\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - g(y)|\,dy$$

$$+ \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} g(y) - g(x) \right|$$

$$+ |g(x) - f(x)|$$

$$\Rightarrow \limsup_{r \to 0+} \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy - f(x) \right|$$

$$\leq M(f-g)(x) + |f(x) - g(x)|$$

$$F_\alpha \overset{\text{def}}{=} \{ M(f-g) > \alpha \}$$

$$G_\alpha \overset{\text{def}}{=} \{ |f-g| > \alpha \}$$

$$\Rightarrow E_\alpha \subset F_\alpha \cup G_\alpha$$

$$m(F_\alpha) \leq \frac{3^n}{\alpha} \|f-g\|_1 < \frac{3^n}{\alpha} \varepsilon$$

(by H-L)

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$$m(G_\alpha) \leq \frac{1}{\alpha} \| f - g \|_1 < \frac{1}{\alpha} \varepsilon$$

(by Tchebyshev)
Chpt 2. Ex. 9

$$\Rightarrow \quad m(E_\alpha) < \frac{3^n + 1}{\alpha} \varepsilon$$

$$\Rightarrow \quad m(E_\alpha) = 0 .$$