第十七讲 (2026.4.29)

Thm (Tonelli)

$\forall f \in L^+(\mathbb{R}^{n_1+n_2})$

(T1) $f^y \in L^+(\mathbb{R}^{n_1})$ for a.e. $y \in \mathbb{R}^{n_2}$

$f_x \in L^+(\mathbb{R}^{n_2})$ for a.e. $x \in \mathbb{R}^{n_1}$

(T2) $y \mapsto \int_{\mathbb{R}^{n_1}} f^y \, dx \in L^+(\mathbb{R}^{n_2})$

$x \mapsto \int_{\mathbb{R}^{n_2}} f_x \, dy \in L^+(\mathbb{R}^{n_1})$

(T3)
$$\int_{\mathbb{R}^{n_1+n_2}} f \, dm = \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} f(x,y) \, dx \right] dy$$
$$= \int_{\mathbb{R}^{n_1}} \left[\int_{\mathbb{R}^{n_2}} f(x,y) \, dy \right] dx$$

Pf 只需证: $\forall E \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$, $\chi_E \in \mathcal{F}$

这里

$$\mathcal{F} \overset{\text{def}}{=} \{ f \in L^+ : f \text{ 满足 } (T1)-(T3) \}$$

Step 1 設 $E = Q' \times Q''$

where $Q' \subset \mathbb{R}^{n_1}$, $Q'' \subset \mathbb{R}^{n_2}$

$$E^y = \begin{cases} Q' & \text{if } y \in Q'' \\ \phi & \text{if } y \notin Q'' \end{cases}$$

$\Rightarrow \forall y \in \mathbb{R}^{n_2}$, $E^y \in \mathcal{L}_{\mathbb{R}^{n_1}}$

$\Rightarrow (\chi_E)^y = \chi_{E^y} \in L^+(\mathbb{R}^{n_1})$

且

$$\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx = m_{n_1}(E^y)$$

$$= \begin{cases} |Q'| & \text{if } y \in Q'' \\ 0 & \text{otherwise} \end{cases}$$

$$= |Q'| \, \chi_{Q'}(y) \in L^+(\mathbb{R}^{n_2})$$

$$\int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \right] dy$$

$$= |Q'| \int_{\mathbb{R}^{n_2}} \chi_{Q''}(y) \, dy$$

$$= |Q'| \, |Q''| = m(E) = \int_{\mathbb{R}^{n_1+n_2}} \chi_E \, dm$$

故 $\chi_E \in \mathcal{F}$

Step 2 $\forall E \overset{\text{open}}{\subset} \mathbb{R}^{n_1+n_2}$

$\exists \{Q_k\}_{k=1}^{\infty}$ 两两不交 s.t.

$$E = \biguplus_{k=1}^{\infty} Q_k \qquad Q_k = Q_k' \times Q_k''$$

$$\Rightarrow \quad E^y = \biguplus_{k=1}^{\infty} (Q_k)^y$$

$$\Rightarrow \quad \underbrace{(\chi_E)^y = \sum_{k=1}^{\infty} \chi_{(Q_k)^y}}_{a.e.} \in L^+(\mathbb{R}^{n_1})$$

$$\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx = m_{n_1}(E^y)$$

$$= \sum_{k=1}^{\infty} m_{n_1}((Q_k)^y)$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}^{n_1}} \chi_{(Q_k)^y} \, dx$$

$$\in L^+(\mathbb{R}^{n_2})$$

$$\int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \right] dy \overset{\text{MCT}}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_{Q_k})^y \, dx \right] dy$$

$$\overset{\text{Step 1}}{=} \sum_{k=1}^{\infty} m(Q_k)$$

④

$$= m(E) = \int_{\mathbb{R}^{n_1+n_2}} \chi_E \, dm$$

Step 3 当 $E \overset{cpt}{\subset} \mathbb{R}^{n_1+n_2}$

$\exists R > 0$, s.t. $E \subset B_R(0)$

令 $G \overset{def}{=} B_R(0) \setminus E$

$\Rightarrow G$ 开

上次课中 Lem 3

$$\Rightarrow \chi_E = \chi_{B_R(0)} - \chi_G \in \mathcal{F}$$

Step 4 当 $E \overset{F_\sigma}{\subset} \mathbb{R}^{n_1+n_2}$

$$E = \bigcup_{k=1}^{\infty} F_k, \quad F_k \text{ 闭}$$

不妨设：$\forall k$. F_k 紧 (否则代以 $F_k \cap \overline{B_k(0)}$)

$$\bigcup_{k=1}^{N} F_k \nearrow E$$

Step 3

$$\Rightarrow \mathcal{F} \ni \chi_{\bigcup_{k=1}^{N} F_k} \nearrow \chi_E$$

Lem 2

$$\Rightarrow \chi_E \in \mathcal{F}$$

Step 5 证 $m(E) = 0$

$\forall k$, $\exists\, G_k$ 开 s.t.

$$\begin{cases} E \subset G_k \\ m(G_k) < \dfrac{1}{k} \end{cases}$$

令

$$G = \bigcap_{k=1}^{\infty} G_k$$

$\Longrightarrow$ G 是 G_δ 集 且 $m(G) = 0$

$$\bigcap_{k=1}^{N} G_k \searrow G$$

$$\Longrightarrow \chi_{\bigcap_{k=1}^{N} G_k} \searrow \chi_G$$

$$\overset{\text{Lem 4}}{\Longrightarrow} \chi_G \in \mathcal{F}$$

$$\Longrightarrow \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_G)^y \, dx \right] dy = \int_{\mathbb{R}^{n_1+n_2}} \chi_G \, dm$$

$$= m(G) = 0$$

$$\Longrightarrow \underbrace{\int_{\mathbb{R}^{n_1}} (\chi_G)^y \, dx}_{m_{n_1}(G^y)} = 0 \quad \text{for a.e. } y \in \mathbb{R}^{n_2}$$

$E^y \subset G^y$ ⑥

$\Longrightarrow \quad m_{n_1}(E^y) = 0 \quad$ for a.e. $y \in \mathbb{R}^{n_2}$

$\Longrightarrow \quad (\chi_E)^y = \chi_{E^y} \in L^+(\mathbb{R}^{n_1})$

for a.e. $y \in \mathbb{R}^{n_2}$

$$\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx = m_{n_1}(E^y) = 0 \in L^+(\mathbb{R}^{n_2})$$

$$\Longrightarrow \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \right] dy = 0 = \int_{\mathbb{R}^{n_1+n_2}} \chi_E \, du$$

故 $\chi_E \in \mathcal{F}$

Step 6 $\quad \forall E \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

$E = F \sqcup Z \quad$ for F_σ 集 F

零集 Z

$\Longrightarrow \quad \chi_E = \chi_F + \chi_Z \in \mathcal{F}$

Cor $\quad \forall E \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

(i) $\quad E^y \in \mathcal{L}_{\mathbb{R}^{n_1}} \quad$ for a.e. $y \in \mathbb{R}^{n_2}$

$E_x \in \mathcal{L}_{\mathbb{R}^{n_2}} \quad$ for a.e. $x \in \mathbb{R}^{n_1}$

(ii) $y \mapsto m_{n_1}(E^y) \in L^+(\mathbb{R}^{n_2})$

$x \mapsto m_{n_2}(E_x) \in L^+(\mathbb{R}^{n_1})$

(iii) $$m(E) = \int_{\mathbb{R}^{n_2}} m_{n_1}(E^y)\,dy = \int_{\mathbb{R}^{n_1}} m_{n_2}(E_x)\,dx$$

(Cavalieri 原理, 祖暅原理)

Prop 1 设 $E_1 \subset \mathbb{R}^{n_1}$, $E_2 \subset \mathbb{R}^{n_2}$

$$\left.\begin{array}{l} E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}} \\ m^*_{n_2}(E_2) > 0 \end{array}\right\} \Rightarrow E_1 \in \mathcal{L}_{\mathbb{R}^{n_1}}$$

Pf

$E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

$\Longleftrightarrow \chi_{E_1 \times E_2} \in L^+(\mathbb{R}^{n_1+n_2})$

Tonelli:

$\Longrightarrow (\chi_{E_1 \times E_2})^y \in L^+(\mathbb{R}^{n_1})$

for a.e. $y \in \mathbb{R}^{n_2}$

而 $$(\chi_{E_1 \times E_2})^y(x) = \chi_{E_1 \times E_2}(x, y) = \chi_{E_1}(x)\,\chi_{E_2}(y)$$

$\Rightarrow$ 只需证： $\exists\, y \in E_2$ s.t. (8)

$$(\chi_{E_1\times E_2})^y \in L^+(\mathbb{R}^{n_1})$$

令 $F \stackrel{def}{=} \{ y \in \mathbb{R}^{n_2} : (E_1 \times E_2)^y \in \mathcal{L}_{\mathbb{R}^{n_1}} \}$

$\xRightarrow{Cor}$ $m_{n_2}(F^c) = 0$

而 $E_2 = (E_2 \cap F) \sqcup (E_2 \cap F^c)$

$\Rightarrow$ $0 < m^*_{n_2}(E_2) \leq m^*_{n_2}(E_2 \cap F) + \underbrace{m^*_{n_2}(E_2 \cap F^c)}_{=0}$

$\Rightarrow$ $m^*_{n_2}(E_2 \cap F) > 0$

$\Rightarrow$ $E_2 \cap F \neq \phi$

<u>Prop 2</u> $\forall\, E_1 \in \mathcal{L}_{\mathbb{R}^{n_1}}$, $\forall\, E_2 \in \mathcal{L}_{\mathbb{R}^{n_2}}$

(i) $E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

(ii) $m_{n_1+n_2}(E_1 \times E_2) = m_{n_1}(E_1)\, m_{n_2}(E_2)$

Lem $\forall E_1 \subset \mathbb{R}^{n_1}, \forall E_2 \subset \mathbb{R}^{n_2}$

$$m^*_{n_1+n_2}(E_1 \times E_2) \leqslant m^*_{n_1}(E_1)\, m^*_{n_2}(E_2)$$

Pf Case 1 $\begin{cases} m^*_{n_1}(E_1) = +\infty \\ m^*_{n_2}(E_2) \neq 0 \end{cases}$

或 $\begin{cases} m^*_{n_1}(E_1) \neq 0 \\ m^*_{n_2}(E_2) = +\infty \end{cases}$

平凡

Case 2 $m^*_{n_1}(E_1),\ m^*_{n_2}(E_2) < +\infty$

$\forall \varepsilon > 0$. $\exists \{Q'_j\}_{j=1}^{\infty} \subset \mathbb{R}^{n_1}$

$\exists \{Q''_k\}_{k=1}^{\infty} \subset \mathbb{R}^{n_2}$

s.t.

$$\begin{cases} E_1 \subset \bigcup_{j=1}^{\infty} Q'_j \\ \sum_{j=1}^{\infty} |Q'_j| < m^*_{n_1}(E_1) + \varepsilon \end{cases}$$

$$\begin{cases} E_2 \subset \bigcup_{k=1}^{\infty} Q''_k \\ \sum_{k=1}^{\infty} |Q''_k| < m^*_{n_2}(E_2) + \varepsilon \end{cases}$$

$\Rightarrow \{Q_j' \times Q_k''\}_{j,k}$ 是 $E_1 \times E_2$ 的

长方体覆盖 (第一章 Ex. 15)

$$\Rightarrow m^*_{n_1+n_2}(E_1 \times E_2) \leq \sum_{j,k} |Q_j' \times Q_k''|$$

$$= \left(\sum_{j=1}^{\infty} |Q_j'|\right)\left(\sum_{k=1}^{\infty} |Q_k''|\right)$$

$$< \left(m^*_{n_1}(E_1) + \varepsilon\right)\left(m^*_{n_2}(E_2) + \varepsilon\right)$$

$$\Rightarrow m^*_{n_1+n_2}(E_1 \times E_2) \leq m^*_{n_1}(E_1)\, m^*_{n_2}(E_2)$$

Case 3 $\begin{cases} m^*_{n_1}(E_1) = +\infty \\ m^*_{n_2}(E_2) = 0 \end{cases}$

或 $\begin{cases} m^*_{n_1}(E_1) = 0 \\ m^*_{n_2}(E_2) = +\infty \end{cases}$

$$E_1 = \bigcup_{k=1}^{\infty} E_1^{(k)} \quad \text{with} \quad E_1^{(k)} = E_1 \cap B_k(0)$$

$$\Rightarrow E_1 \times E_2 = \bigcup_{k=1}^{\infty} (E_1^{(k)} \times E_2)$$

$$\Rightarrow m^*_{n_1+n_2}(E_1 \times E_2) \leq \sum_{k=1}^{\infty} m^*_{n_1+n_2}(E_1^{(k)} \times E_2)$$

$$\overset{\text{Case 2}}{\leq} \sum_{k=1}^{\infty} m^*_{n_1}(E_1)\, m^*_{n_2}(E_2) = 0$$

$$= \text{RHS}$$

Pf of Prop 2

- $E_1 \in \mathcal{L}_{\mathbb{R}^{n_1}} \iff \exists\, G_1 \overset{G_\delta}{\subset} \mathbb{R}^{n_1}$

  $\exists\, Z_1 \subset \mathbb{R}^{n_1},\ m_{n_1}(Z_1) = 0$

  s.t. $E_1 = G_1 \setminus Z_1$

  $E_2 \in \mathcal{L}_{\mathbb{R}^{n_2}} \iff E_2 = \underset{\uparrow \atop G_\delta\text{集}}{G_2} \setminus \underset{\uparrow \atop \text{零集}}{Z_2}$

- $\Rightarrow\ G_1 \times G_2 \overset{G_\delta}{\subset} \mathbb{R}^{n_1+n_2}$

  $(G_1 \times G_2) \setminus (E_1 \times E_2)$

  $\subset (\underbrace{Z_1 \times G_2}_{\text{零集}}) \cup (\underbrace{G_1 \times Z_2}_{\text{零集}})$ (by Lem)

- $\Rightarrow\ m^*_{n_1+n_2}((G_1 \times G_2) \setminus (E_1 \times E_2)) = 0$

  $\Rightarrow\ E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

  等式 $m_{n_1+n_2}(E_1 \times E_2) = m_{n_1}(E_1)\, m_{n_2}(E_2)$

  由 Cor 立见

HW: Ex. 14. 17. 20

HW: $\forall f \in L^p$

$$\|f\|_p^p = p \int_0^\infty m(\{|f| > t\})\, t^{p-1}\, dt$$