第十四讲（2026.4.20）

Thm　$L^\infty(E)$ 完备.

Pf：设 $\{f_k\}_{k=1}^\infty$ 是 $L^\infty(E)$ 中 Cauchy 列

$\Rightarrow \forall \varepsilon>0.\ \exists N$ s.t.

$$\|f_k-f_j\|_\infty<\varepsilon,\quad \forall k.j\ge N$$

令

$$A_k \overset{def}{=} \{|f_k|>\|f_k\|_\infty\}$$

$$B_{k,j} \overset{def}{=} \{|f_k-f_j|>\|f_k-f_j\|_\infty\}$$

$$k,j=1.2,\cdots$$

$\Rightarrow\ m(A_k)=m(B_{k,j})=0,\quad k.j=1,2,\cdots$

令

$$F \overset{def}{=} \Big(\bigcup_k A_k\Big)\cup\Big(\bigcup_{k,j} B_{k,j}\Big)$$

$\Rightarrow\ m(F)=0$

$\forall x\in E\setminus F,$

$$|f_k(x)-f_j(x)|\le\|f_k-f_j\|_\infty$$

$\Rightarrow \{f_k(x)\}_{k=1}^{\infty}$ 是 $\mathbb{R}$ 中 Cauchy 列.

令
$$f(x) \overset{\text{def}}{=} \begin{cases} \lim\limits_{k\to\infty} f_k(x), & x \in E \setminus F \\ 0, & x \in F \end{cases}$$

$\Rightarrow f$ 可测

$\exists N$ s.t.

$$\sup_{x\in E\setminus F} |f_k(x) - f_j(x)| \leq \|f_k - f_j\|_\infty \leq 1 \qquad \forall k, j \geq N$$

$$\overset{j\to\infty}{\Longrightarrow} \sup_{x\in E\setminus F} |f_N(x) - f(x)| \leq 1.$$

$$\Rightarrow \sup_{x\in E\setminus F} |f(x)| \leq 1 + \sup_{x\in E\setminus F} |f_N(x)| < +\infty$$

$$\Rightarrow f \in L^\infty(E)$$

$\forall \varepsilon > 0$, $\exists N$ s.t.

$$\|f_k - f_j\|_\infty < \varepsilon, \qquad \forall k, j \geq N$$

$\Rightarrow \forall x \in E\setminus F$, $\forall k \geq N$

$$|f_k(x) - f(x)| = \lim_{j\to\infty} |f_k(x) - f_j(x)|$$
$$\leq \lim_{j\to\infty} \|f_k - f_j\|_\infty \leq \varepsilon$$

$$\Rightarrow \quad \|f_k - f\|_\infty = \inf_{\substack{Z\subset E \\ m(Z)=0}} \sup_{x\in E\setminus Z} |f_k(x) - f(x)|$$

$$\leq \varepsilon, \quad \forall k \geq N$$

Def 设 $1 \leq p < \infty$. 对 $\{f_k\}_{k=1}^\infty \subset L^p(E)$,
如果 $\exists f \in L^p(E)$ s.t.

$$\|f_k - f\|_p \to 0 \quad \text{as } k \to \infty$$

则称 f_k 按 L^p 范数收敛于 f, 记为
$f_k \xrightarrow{L^p} f$.

Def 对 E 上可测函数列 $\{f_k\}_{k=1}^\infty$,
如果 $\exists f$ s.t. $\forall \varepsilon > 0$,

$$m(\{|f_k - f| \geq \varepsilon\}) \to 0 \quad \text{as } k \to \infty$$

则称 f_k 依测度收敛于 f, 记为 $f_k \xrightarrow{m} f$

Prop 设 $1 \leq p < \infty$.

$$f_k \xrightarrow{L^p} f \quad \Longrightarrow \quad f_k \xrightarrow{m} f$$

Pf $\forall \varepsilon > 0$.

$$\int_E |f_k - f|^p \, dm \geq \int_{\{|f_k - f| \geq \varepsilon\}} |f_k - f|^p \, dm$$

$$\geq \varepsilon^p \, m(\{|f_k - f| \geq \varepsilon\})$$

$$\Rightarrow \quad m(\{|f_k - f| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^p} \|f_k - f\|_p^p$$

$$\to 0 \quad \text{as } k \to \infty$$

Rmk: 1° $f_k \xrightarrow{m} f \not\Rightarrow f_k \to f$ a.e.

(反例: Ex. 12)

2° $f_k \to f$ a.e. $\not\Rightarrow f_k \xrightarrow{m} f$

反例: $f_k = \chi_{(-k, k)}$, $f \equiv 1$.

$\Rightarrow f_k \to f$ pointwise

但 $m(\{|f_k - f| \geq \frac{1}{2}\}) = +\infty$

Thm (Lebesgue)

设 $m(E) < +\infty$, f, f_k, $k = 1, 2, \dots$ 在 E 上可测 且 a.e. 有限.

$$f_k \to f \text{ a.e.} \implies f_k \xrightarrow{m} f$$

Pf 对 $k \in \mathbb{N}$, $\varepsilon > 0$

$$E_k(\varepsilon) \overset{\text{def}}{=} \{ |f_k - f| \geq \varepsilon \}.$$

要证明: $\forall \varepsilon > 0$.

$$m(E_k(\varepsilon)) \to 0 \quad \text{as} \quad k \to \infty$$

$$\underbrace{\forall x \in \limsup_{k \to \infty} E_k(\varepsilon)}_{} \overset{\text{def}}{=} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k(\varepsilon)$$

$$\forall j, \ \exists k_j > j \quad \text{s.t.} \quad x \in E_{k_j}(\varepsilon)$$

$\implies$ 存在子列 $\{f_{k_j}\}_{k=1}^{\infty}$ s.t.

$$|f_{k_j}(x) - f(x)| \geq \varepsilon$$

$$\implies f_k(x) \not\to f(x)$$

⑥

$$\Longrightarrow \limsup_{k\to\infty} E_k(\varepsilon) \subset \{f_k \not\to f\}$$

$f_k \to f$ a.e.

$$\Longrightarrow \quad m\left(\limsup_{k\to\infty} E_k(\varepsilon)\right) = 0$$

$$\text{(ii)} \quad \bigcup_{k=j}^{\infty} E_k(\varepsilon) \searrow \limsup_{k\to\infty} E_k(\varepsilon)$$

$$\underset{m(E)<+\infty}{\overset{\text{测度的连续性}}{\Longrightarrow}} \quad \lim_{j\to\infty}\left(\bigcup_{k=j}^{\infty} E_k(\varepsilon)\right) = 0$$

$$\Longrightarrow \quad \lim_{k\to\infty} m(E_k(\varepsilon)) = 0 .$$

Thm (Riesz)

$$f_k \xrightarrow{m} f \quad \Longrightarrow \quad \exists \text{ 子列 } f_{k_j} \to f \text{ a.e.}$$

Pf $\quad f_k \xrightarrow{m} f$

$$\overset{\text{def}}{\Longleftrightarrow} \quad \forall \varepsilon > 0,\ \forall \eta > 0,\ \exists N \text{ s.t.}$$

$$m(\{|f_k - f| \geq \varepsilon\}) < \eta, \quad \forall k \geq N.$$

$\varepsilon = \eta = \frac{1}{2^j}$

$\Longrightarrow \quad \forall j, \quad \exists\, k_j > k_{j-1} \quad$ s.t.

$$m(\{ |f_k - f| \geq \frac{1}{2^j} \}) < \frac{1}{2^j}, \quad \forall k \geq k_j$$

$\Longrightarrow$ ∃ 子列 $\{f_{k_j}\}_{j=1}^{\infty}$ s.t.

$$m(\{ |f_{k_j} - f| \geq \frac{1}{2^j} \}) < \frac{1}{2^j}, \quad j = 1, 2, \cdots$$

令

$$E_j \overset{\text{def}}{=} \{ |f_{k_j} - f| \geq \frac{1}{2^j} \}$$

$$F_N \overset{\text{def}}{=} \bigcap_{j=N}^{\infty} (E \setminus E_j)$$

$\Longrightarrow \quad \forall x \in F_N$

$$|f_{k_j}(x) - f(x)| < \frac{1}{2^j}, \quad j \geq N$$

$\Longrightarrow \quad f_{k_j} \to f$ on F_N

令

$$F \overset{\text{def}}{=} \bigcup_{N=1}^{\infty} F_N = \liminf_{j \to \infty} (E \setminus E_j)$$

$\Longrightarrow \quad f_{k_j} \to f$ on F.

<u>Claim</u> $m(E \setminus F) = 0$.

$$E \setminus F = \bigcap_{N=1}^{\infty} (E \setminus F_N)$$

$$= \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} E_j = \limsup_{j \to \infty} E_j$$

而

$$\sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < +\infty$$

Borel - Cantelli

$$\Longrightarrow \quad m(\limsup_{j \to \infty} E_j) = 0.$$

L^p 范数收敛

pointwise 收敛

$\Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Downarrow$

a.e. 收敛 $\xRightarrow{\text{如果 } m(E)<\infty}$ 依测度收敛.

依测度收敛 $\xrightarrow{\text{有子列}}$ a.e. 收敛

如果 $m(E)<\infty$ $\Downarrow$ Egorov

$\underbrace{\text{近一致收敛}}$

$\forall \varepsilon > 0$, $\exists A \subset E$ s.t. $\begin{cases} m(A) < \varepsilon \\ f_k \rightrightarrows f \text{ on } E \setminus A \end{cases}$

HW: 1° 研究依 L^∞ 范数收敛 (i.e. $\|f_k - f\|_\infty \to 0$ as $k \to \infty$) 与近一致收敛的关系

2° 研究近一致收敛与 a.e. 收敛的关系，与依测度收敛的关系