第十二讲 (2026.4.13)

Thm (积分的绝对连续性)

$f \in L^1 \Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$\int_E |f|\,dm < \varepsilon, \quad \forall E \in \mathcal{L} \text{ with } m(E) < \delta.$$

Pf: 令

$$E_k \overset{\text{def}}{=} \{|f| \le k\}$$

$$f_k \overset{\text{def}}{=} |f| \cdot \chi_{E_k}, \qquad k = 1, 2, \cdots$$

$\Rightarrow f_k \nearrow |f|$

$$\overset{\text{MCT}}{\Longrightarrow} \lim_{k\to\infty} \int f_k\,dm = \int |f|\,dm$$

$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t.

$$0 \le \int |f|\,dm - \int f_k\,dm < \varepsilon/2$$

令 $\delta \overset{\text{def}}{=} \dfrac{\varepsilon}{2N}$

$\Rightarrow \forall E \in \mathcal{L}$ with $m(E) < \delta$,

$$\int_E |f|\,dm = \int_E (|f| - f_N)\,dm + \int f_N\,dm$$

$$< \varepsilon/2 + N \cdot m(E) < \varepsilon$$

Thm (Lebesgue控制收敛定理, DCT)

$$\begin{cases} f_k \to f \quad \text{a.e.} \\ \exists\, g \in L^1 \quad \text{s.t.} \quad |f_k| \le g \quad \text{a.e.} \end{cases}$$

$$\Rightarrow \lim_{k\to\infty} \int f_k \, dm = \int f \, dm$$

Pf.

$$\left.\begin{array}{l} f_k \to f \quad \text{a.e.} \\ |f_k| \le g \quad \text{a.e.} \end{array}\right\} \Rightarrow |f| \le g \quad \text{a.e.}$$

$$\Rightarrow \int |f| \, dm \le \int g \, dm < +\infty$$

$$\Rightarrow f \in L^1$$

今

$$g_k \overset{\text{def}}{=} |f_k - f|$$

$$\Rightarrow 0 \le g_k \le 2g \quad \text{a.e.} \quad k = 1, 2, \cdots$$

由 Fatou,

$$\int \liminf_{k\to\infty} (2g - g_k) \, dm \le \liminf_{k\to\infty} \int (2g - g_k) \, dm$$

$$\Rightarrow \cancel{2\int g\,dm} - \int \lim_{k\to\infty} g_k\,dm$$

$$\leq \cancel{2\int g\,dm} - \limsup_{k\to\infty} \int g_k\,dm$$

$$\Rightarrow \limsup_{k\to\infty} \int g_k\,dm \leq \int \lim_{k\to\infty} g_k\,dm$$

$$= 0$$

$$\Rightarrow \lim_{k\to\infty} \int |f_k - f|\,dm = 0$$

$$\Rightarrow \lim_{k\to\infty} \int f_k\,dm = \int f\,dm$$

<u>Cor</u> (有界收斂定理)

設 f, $\{f_k\}_{k=1}^{\infty}$ 可測, s.t.

(i) $\exists M > 0$ s.t. $|f_k| \leq M$ a.e.

(ii) $\exists E \in \mathcal{L}$ with $m(E) < \infty$ s.t.

$$\operatorname{supp}(f_k) \subset E, \quad \forall k$$

(iii) $f_k \to f$ a.e.

則 $$\lim_{k\to\infty} \int f_k\,dm = \int f\,dm$$

Pf 令 $g \overset{\text{def}}{=} M \cdot \chi_E$.

例：求 $\displaystyle\lim_{k\to\infty} \int_0^\infty \frac{dt}{\left(1+\frac{t}{k}\right)^k t^{1/k}}$

Note $\displaystyle\frac{1}{\left(1+\frac{t}{k}\right)^k t^{1/k}} \to e^{-t}$ as $k\to\infty$

$$\overset{?}{\Rightarrow} \lim_{k\to\infty} \int_0^\infty \frac{dt}{\left(1+\frac{t}{k}\right)^k t^{1/k}} = \int_0^\infty e^{-t}\,dt$$

$$= -e^{-t}\Big|_0^\infty = 1.$$

(i) 当 $t\in[0,1]$，$k\geq 2$ 时

$$\frac{1}{\left(1+\frac{t}{k}\right)^k t^{1/k}} \leq \frac{1}{\sqrt{t}} \in L^1[0,1]$$

(ii) 当 $t\in[1,+\infty)$，$k\geq 2$ 时

$$\frac{1}{\left(1+\frac{t}{k}\right)^k t^{1/k}} \leq \frac{4}{t^2} \in L^1[1,+\infty).$$

$$\left[\left(1+\frac{t}{k}\right)^k \geq \binom{k}{2} \cdot \left(\frac{t}{k}\right)^2 = \frac{k(k-1)}{2} \cdot \frac{t^2}{k^2} \geq \frac{t^2}{4} \right]$$

令 $$g(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \text{if } t \in (0, 1] \\ \frac{4}{t^2}, & \text{if } t \in (1, +\infty) \end{cases}$$ ⑤

$\Rightarrow$ g 是可积控制函数.

Thm (积分号下求导)

设 $E \in \mathcal{L}(\mathbb{R}^n)$. 函数 $f: E \times (a, b) \to \mathbb{R}$

s.t.

(i) $\forall y \in (a, b)$, $f(\cdot, y) \in L^1(E)$

(ii) $\forall x \in E$, $y \mapsto f(x, y)$ 在 (a, b) 上可微

(iii) $\exists g \in L^1(E)$ s.t.

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq g(x), \quad \forall (x, y) \in E \times (a, b)$$

则

$$\frac{\partial}{\partial y} \int_E f(x, y)\, dx = \int_E \frac{\partial f}{\partial y}(x, y)\, dx$$

⑥

Pf 固定 (x, y). $\forall t_k \to 0$ with $y + t_k \in (a, b)$, 令

$$f_k(x) \overset{def}{=} \frac{f(x, y+t_k) - f(x, y)}{t_k}$$

$$\Rightarrow \quad f_k(x) \to \frac{\partial f}{\partial y}(x, y)$$

且

$$|f_k(x)| \le \sup_{y \in (a,b)} \left| \frac{\partial f}{\partial y}(x, y) \right| \le g(x)$$

(by Lagrange 定理 (iii))

$$\overset{DCT}{\Longrightarrow} \int_E \frac{\partial f}{\partial y}(x, y)\, dx = \lim_{k \to \infty} \int_E f_k(x)\, dx$$

$$= \lim_{k \to \infty} \frac{\int_E f(x, y+t_k)\, dx - \int_E f(x, y)\, dx}{t_k}$$

$$= \frac{\partial}{\partial y} \int_E f(x, y)\, dx$$

复值函数的积分

Def 对函数 $f: E \to \mathbb{C}$，如果 $\mathrm{Re} f$，$\mathrm{Im} f$ 都可测，则称 f 可测.

如果

$$\int_E |f|\, dm < +\infty$$

则称 f 在 E 上可积. 并令

$$\int_E f\, dm \overset{\text{def}}{=} \int_E \mathrm{Re} f\, dm + i \int_E \mathrm{Im} f\, dm$$

Lebesgue 积分与 Riemann 积分的关系.

Thm 对函数 $f: [a, b] \to \mathbb{R}$

Riemann 可积 $\Rightarrow$ Lebesgue 可积

且

$$\int_{[a,b]} f\, dm = \int_a^b f(x)\, dx$$

Pf 对 $[a, b]$ 的划分

$$P: \quad a = x_0 < x_1 < \cdots < x_n = b$$

$$S(f,P) \overset{\text{def}}{=} \sum_{k=1}^{n} M_k (x_k - x_{k-1}) \quad (\text{Darboux 上和})$$

$$s(f,P) \overset{\text{def}}{=} \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \quad (\text{Darboux 下和})$$

其中

$$M_k = \sup_{x\in[x_{k-1},x_k]} f(x)$$

$$m_k = \inf_{x\in[x_{k-1},x_k]} f(x)$$

$$\overline{\int_a^b} f \overset{\text{def}}{=} \inf_P S(f,P) \quad (\text{上积分})$$

$$\underline{\int_a^b} f \overset{\text{def}}{=} \sup_P s(f,P). \quad (\text{下积分})$$

f Riemann 可积 $\iff \overline{\int_a^b} f = \underline{\int_a^b} f$

$\exists$ 单调划分序列 $\{P_k\}_{k=1}^{\infty}$ s.t.

$$S(f,P_k) \searrow \overline{\int_a^b} f$$

$$s(f,P_k) \nearrow \underline{\int_a^b} f$$

设 $P_k: \ a = x_0^{(k)} < x_1^{(k)} < \cdots < x_{n_k}^{(k)} = b$

$$\varphi_k \overset{\text{def}}{=} \sum_{j=1}^{n_k} M_j^{(k)} \chi_{(x_{j-1}^{(k)},\, x_j^{(k)}]}$$

$$\psi_k \overset{\text{def}}{=} \sum_{j=1}^{n_k} m_j^{(k)} \chi_{(x_{j-1}^{(k)},\, x_j^{(k)}]}$$

$$\Longrightarrow \quad \varphi_k \searrow, \quad \psi_k \nearrow$$

(1)

$$\psi_k \le f \le \varphi_k$$

$$g \overset{\text{def}}{=} \lim_{k \to \infty} \psi_k$$

$$h \overset{\text{def}}{=} \lim_{k \to \infty} \varphi_k$$

$$\Longrightarrow \quad g \le f \le h$$

また $|f| \le M$ $\quad$ ($\because$ f Riemann 可積分)

$$\Longrightarrow \quad |\varphi_k| \le M, \quad |\psi_k| \le M$$

DCT

$$\Longrightarrow \quad \int_{[a,b]} |g|\, dm = \lim_{k \to \infty} \int |\psi_k|\, dm \le M(b-a)$$

$$\Longrightarrow \quad g \in L^1[a,b]$$

(2)

$$\int_{[a,b]} g \, dm \overset{DCT}{=} \lim_{k\to\infty} \int \varphi_k \, dm$$

$$= \lim_{k\to\infty} s(f, P_k) = \underline{\int_a^b} f$$

[3]より,

$$\int_{[a,b]} h \, dm = \overline{\int_a^b} f$$

f Riemann 可積分 $\iff \overline{\int_a^b} f = \underline{\int_a^b} f$

$$\iff \int_{[a,b]} h \, dm = \int_{[a,b]} g \, dm$$

$$\iff \int_{[a,b]} (\underbrace{h-g}_{\geq 0}) \, dm = 0$$

$$\iff g = h \quad \text{a.e.}$$

$$\implies f = g \quad \text{a.e.}$$

$$\overset{g \in L^1[a,b]}{\implies} f \in L^1[a,b]$$

以

$$\int_{[a,b]} f \, dm = \int_{[a,b]} g \, dm = \underline{\int_a^b} f = \int_a^b f(x)\,dx$$

Rmk: Lebesgue 积分理论不能处理条件收敛的广义积分.

例. $f(x) = \frac{\sin x}{x}$

则 $f \notin L^1(\mathbb{R})$ (HW)

但 $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$

HW: Ex. 11. 12. 15

1° 计算 $\lim_{n\to\infty} \int_1^2 \frac{n^2 \sin(\frac{x}{n})}{1+nx^2} dx$

2° 设 $E \in \mathcal{L}(\mathbb{R}^n)$, $f \in L^1(E)$. $f \geq 0$

证明: $\lim_{n\to\infty} \int_E n \ln\left(1 + \frac{f(x)}{n}\right) dx = \int_E f \, dm$