第十讲 (2026.4.1)

积分论

Def 对非负简单函数

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} \quad \text{with} \quad \bigsqcup_{k=1}^{N} E_k = \mathbb{R}^n$$

定义

$$\int_{\mathbb{R}^n} \varphi \, dm \overset{\text{def}}{=} \sum_{k=1}^{N} a_k m(E_k) \quad (\text{可能为} +\infty)$$

称为 φ 在 $\mathbb{R}^n$ 上的 (Lebesgue) 积分.

设 $E \in \mathcal{L}$

$$\int_E \varphi \, dm \overset{\text{def}}{=} \int_{\mathbb{R}^n} \varphi \chi_E \, dm = \sum_{k=1}^{N} a_k m(E_k \cap E)$$

Rmk: (良定)

如果 $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} = \sum_{j=1}^{M} b_j \chi_{F_j}$

with $\bigsqcup_{k=1}^{N} E_k = \mathbb{R}^n = \bigsqcup_{j=1}^{M} F_j$

则 $$\sum_{k=1}^{N} a_k m(E_k) = \sum_{j=1}^{M} b_j m(F_j)$$

②

$$E_k = \bigsqcup_{j=1}^{M} (E_k \cap F_j)$$

$$F_j = \bigsqcup_{k=1}^{N} (E_k \cap F_j)$$

可加性

$$\Longrightarrow \quad m(E_k) = \sum_{j=1}^{M} m(E_k \cap F_j) = m(F_j)$$

而且

$$E_k \cap F_j \neq \phi \quad \text{implies} \quad a_k = b_j$$

$$\Rightarrow \quad \sum_{k=1}^{N} a_k \, m(E_k) = \sum_{k=1}^{N} a_k \sum_{j=1}^{M} m(E_k \cap F_j)$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{M} b_j \, m(E_k \cap F_j)$$

$$= \sum_{j=1}^{M} b_j \, m(F_j)$$

例： $f = \chi_{\mathbb{Q}}$

$$\int_{\mathbb{R}} f \, dm = 1 \cdot \underbrace{m(\mathbb{Q})}_{0} + 0 \cdot \underbrace{m(\mathbb{R} \setminus \mathbb{Q})}_{+\infty} = 0$$

例：

$$\int_{\mathbb{R}^n} \chi_E \, dm = m(E)$$

$$\int_E \chi_F \, dm = m(E \cap F)$$

Prop ∀ 非负简单函数 φ, ψ, $\forall \alpha, \beta \geq 0$,

$$\int (\alpha\varphi + \beta\psi)\, dm = \alpha \int \varphi\, dm + \beta \int \psi\, dm$$

Pf $\int \alpha\varphi\, dm = \alpha \int \varphi\, dm$ 平凡.

求证明:
$$\int (\varphi + \psi)\, dm = \int \varphi\, dm + \int \psi\, dm$$

设
$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} \quad \text{with} \quad \bigsqcup_{k=1}^{N} E_k = \mathbb{R}^n$$

$$\psi = \sum_{j=1}^{N} b_j \chi_{F_j} \quad \text{with} \quad \bigsqcup_{j=1}^{M} F_j = \mathbb{R}^n$$

$$\Rightarrow \quad E_k = \bigsqcup_{j=1}^{M} (E_k \cap F_j)$$

$$F_j = \bigsqcup_{k=1}^{N} (E_k \cap F_j)$$

$$\Rightarrow \quad \varphi + \psi = \sum_{k=1}^{N} \sum_{j=1}^{M} (a_k + b_j) \chi_{E_k \cap F_j}$$

$$\Rightarrow \quad \int (\varphi + \psi)\, dm = \sum_{k=1}^{N} \sum_{j=1}^{M} (a_k + b_j)\, m(E_k \cap F_j)$$

$$= \sum_{k=1}^{N} a_k \sum_{j=1}^{M} m(E_k \cap F_j)$$

$$+ \sum_{j=1}^{M} b_j \sum_{k=1}^{N} m(E_k \cap F_j)$$

④

$$= \sum_{k=1}^{N} a_k m(E_k) + \sum_{j=1}^{M} b_j m(F_j)$$

$$= \int \varphi \, dm + \int \psi \, dm$$

Prop（可加性）

$\forall$ 非负简单 φ. $\forall E_1, E_2 \in \mathcal{L}$. $E_1 \cap E_2 = \phi$

$$\int_{E_1 \sqcup E_2} \varphi \, dm = \int_{E_1} \varphi \, dm + \int_{E_2} \varphi \, dm$$

Pf

$$\text{LHS} = \int_{\mathbb{R}^n} \varphi \chi_{E_1 \sqcup E_2} dm$$

$$= \int_{\mathbb{R}^n} \varphi (\chi_{E_1} + \chi_{E_2}) \, dm$$

$$= \int_{\mathbb{R}^n} \varphi \chi_{E_1} dm + \int_{\mathbb{R}^n} \varphi \chi_{E_2} dm$$

$$= \int_{E_1} \varphi \, dm + \int_{E_2} \varphi \, dm$$

Prop（单调性）$\forall$ 非负简单 $\varphi. \psi$

$$\varphi \leq \psi \Rightarrow \int \varphi \, dm \leq \int \psi \, dm$$

Pf 设

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} \quad \text{with} \quad \bigsqcup_{k=1}^{N} E_k = \mathbb{R}^n$$

$$\psi = \sum_{j=1}^{M} b_j \chi_{F_j} \quad \text{with} \quad \bigsqcup_{j=1}^{M} F_j = \mathbb{R}^n$$

$$\varphi \leq \psi \quad \Longrightarrow \quad a_k \leq b_j \quad \text{if} \quad E_k \cap F_j \neq \phi$$

$$\Longrightarrow \int \varphi \, dm = \sum_{k,j} a_k \, m(E_k \cap F_j)$$

$$\leq \sum_{k,j} b_j \, m(E_k \cap F_j)$$

$$= \int \psi \, dm$$

Def 对 $\mathbb{R}^n$ 上非负可测函数

$$\int_{\mathbb{R}^n} f \, dm = \int_{\mathbb{R}^n} f(x) \, dx$$

$$\stackrel{\text{def}}{=} \sup \left\{ \int_{\mathbb{R}^n} \varphi \, dm : \varphi \text{ 简单 with } 0 \leq \varphi \leq f \right\}$$

称为 f 在 $\mathbb{R}^n$ 上的积分

如果 $\int_{\mathbb{R}^n} f \, dm < +\infty$，则称 f (Lebesgue) 可积.

如果 f 在 E 上非负可测

$$\int_E f\,dm \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f\chi_E\,dm$$

如果 $\int_E f\,dm < +\infty$，则称 f 在 E 上

可积。

Prop（单调性）设 f, g 非负可测，

$$f \le g \Rightarrow \int f\,dm \le \int g\,dm$$

Pf $\forall\ \varphi$ 简单 with $0 \le \varphi \le f$

$$\stackrel{f\le g}{\Longrightarrow} \quad \varphi \le g$$

$$\Rightarrow \int \varphi\,dm \le \int g\,dm \quad \left(\text{由} \int g\,dm \text{ 的定义}\right)$$

$$\Rightarrow \int f\,dm \le \int g\,dm$$

$\left(\text{由} \int f\,dm \text{ 的定义}\right.$

Prop 对 E 上非负可测函数

$$\int_E f\,dm = 0 \Rightarrow f = 0 \text{ a.e. on } E$$

Pf "$\Leftarrow$" 平凡

"$\Rightarrow$"

$\forall k$，令

$$E_k \overset{\text{def}}{=} \left\{ f > \frac{1}{k} \right\}$$

$$\Rightarrow \quad \frac{1}{k} m(E_k) = \int_{E_k} \frac{1}{k}\, dm$$

$$\leq \int_{E_k} f\, dm \leq \int_E f\, dm = 0$$

$$\Rightarrow \quad m(E_k) = 0, \quad \forall k$$

又 $E_k \nearrow \{f > 0\}$

$\overset{\text{测度的连续性}}{\Longrightarrow} \quad m(\{f>0\}) = 0$

Cor 改变 f 在一零测集上的取值，不改变其积分.

Prop $\left.\begin{array}{l} f \in L^1(E) \\ f \geq 0 \end{array}\right\} \Rightarrow f$ 在 E 上 a.e. 有限.

Pf $\forall k$，令

$$E_k \overset{def}{=} \{f > k\}$$

$$\Rightarrow \quad E_k \searrow \{f = +\infty\}$$

$$k\,m(E_k) = \int_{E_k} k\,dm \leq \int_E f\,dm < +\infty$$

$$\Rightarrow \quad m(E_k) \leq \frac{1}{k}\int_E f\,dm$$

$$\Rightarrow \quad \lim_{k\to\infty} m(E_k) = 0$$

$$\begin{cases} m(E_1) \leq \int_E f\,dm < +\infty \\ E_k \searrow \{f = +\infty\} \end{cases}$$

测度的连续性
$$\Rightarrow \quad m(\{f = +\infty\}) = \lim_{k\to\infty} m(E_k) = 0$$

<u>Thm</u>（Levi 单调收敛定理，MCT）

设 $\{f_k\}_{k=1}^{\infty}$ 是 E 上非负可测函数列 s.t.

$$f_k \nearrow f \quad \text{a.e. on } E$$

则

$$\lim_{k\to\infty}\int_E f_k\,dm = \int_E f\,dm$$

⑨

Pf 不妨设 $f_k \nearrow f$ pointwise

（否则修改 f_k 在零测集上的取值）

积分的单调性

$$\Longrightarrow \int_E f_k \, dm \leq \int_E f_{k+1} \, dm, \quad \forall k$$

$$\Rightarrow \lim_{k\to\infty} \int_E f_k \, dm \text{ 存在（可能为 } +\infty\text{）}$$

Case 1 : $\lim_{k\to\infty} \int_E f_k \, dm = +\infty$

$$f_k \leq f \Longrightarrow \int_E f \, dm \geq \int_E f_k \, dm \to +\infty$$

$$\Longrightarrow \int_E f \, dm = +\infty$$

Case 2 $\lim_{k\to\infty} \int_E f_k \, dm < +\infty$

$$f_k \leq f \Longrightarrow \lim_{k\to\infty} \int_E f_k \, dm \leq \int_E f \, dm$$

Claim: $\lim_{k\to\infty} \int_E f_k \, dm \geq \int_E f \, dm$

$\forall$ 簡単函数 φ with $0 \leq \varphi \leq f$ (10)

$\forall\ \alpha \in (0, 1)$

令

$$E_k \overset{\mathrm{def}}{=} \{ f_k \geq \alpha \varphi \}, \qquad k = 1, 2, \cdots$$

$$\Longrightarrow \quad E_k \nearrow E$$

令

$$\varphi = \sum_{k=1}^{N} a_j \chi_{F_j} \quad \text{with} \quad \bigsqcup_{j=1}^{N} F_j = \mathbb{R}^n$$

$$\xrightarrow{E_k \cap F_j \nearrow F_j} \quad \sum_{j=1}^{N} a_j\, m(E_k \cap F_j) \longrightarrow \sum_{j=1}^{N} a_j\, m(F_j)$$

$$(\text{as } k \to \infty)$$

$$\Longleftrightarrow \quad \int_{E_k} \varphi\, dm \longrightarrow \int_E \varphi\, dm \quad \text{as } k \to \infty$$

$$\int_E f_k\, dm \geq \int_{E_k} f_k\, dm$$

$$\geq \int_{E_k} \alpha \varphi\, dm = \alpha \int_{E_k} \varphi\, dm$$

$$\Longrightarrow \quad \lim_{k \to \infty} \int_E f_k\, dm \geq \alpha \lim_{k \to \infty} \int_{E_k} \varphi\, dm$$

$$= \alpha \int_E \varphi\, dm$$

$$\xrightarrow{\alpha \to 1^-} \quad \lim_{k \to \infty} \int_E f_k\, dm \geq \int_E \varphi\, dm$$

$$\Longrightarrow \quad \lim_{k \to \infty} \int_E f_k\, dm \geq \int_E f\, dm$$