第八讲 (2026.3.25)

Thm $\forall$ 非负可测函数 f,

$\exists$ 非负简单函数 $\{\varphi_k\}_{k=1}^{\infty}$ s.t.

$$\varphi_k \nearrow f$$

i.e. $\forall x \in \mathbb{R}^n$,

$$\begin{cases} 0 \leq \varphi_1(x) \leq \varphi_2(x) \leq \cdots \leq f(x) \\ \varphi_k(x) \to f(x) \quad \text{as} \quad k \to \infty \end{cases}$$

进而,如果 f 有界,则 $\varphi_k \rightrightarrows f$

Pf 对 $k = 1, 2, \cdots$

$j = 0, 1, 2, \cdots, 2^{2k} - 1$

$$E_{k,j} \overset{\text{def}}{=} \left\{ \frac{j}{2^k} \leq f < \frac{j+1}{2^k} \right\}$$

$$F_k \overset{\text{def}}{=} \left\{ f \geq 2^k \right\}$$

$$\varphi_k \overset{\text{def}}{=} \sum_{j=0}^{2^{2k}-1} \frac{j}{2^k} \chi_{E_{kj}} + 2^k \chi_{F_k}$$

Note: $\mathbb{R}^n = \bigsqcup_{j=0}^{2^{2k}-1} E_{k,j} \sqcup F_k$

$$\varphi_k(x) = \begin{cases} \frac{j}{2^k} & \text{if } x \in E_{k,j} \\ 2^k & \text{if } x \in F_k \end{cases}$$

$\Rightarrow \quad 0 \leq \varphi_k \leq f$

Step 1 $\quad \varphi_k \leq \varphi_{k+1}$

Case 1 $\quad x \in F_k$

(i) Subcase 1 $\quad x \in F_{k+1}$

$$\varphi_{k+1}(x) = 2^{k+1} > 2^k = \varphi_k(x)$$

(ii) Subcase 2 $\quad x \in F_k \setminus F_{k+1}$

$$F_k \setminus F_{k+1} = \{ 2^k \leq f < 2^{k+1} \}$$

$$= \bigsqcup_{j=2^{2k+1}}^{2^{2k+2}-1} E_{k+1,j}$$

$$\Rightarrow \quad \varphi_{k+1}(x) \geq \frac{2^{2k+1}}{2^{k+1}} = 2^k = \varphi_k(x)$$

Case 2 $\quad x \notin F_k$

$$\exists \, j \in \{0, 1, \dots, 2^{2k}-1\} \text{ s.t. } x \in E_{k,j}$$

$$E_{k,j} = E_{k+1,2j} \sqcup E_{k+1,2j+1}.$$

$$\Rightarrow \quad \varphi_{k+1}(x) \geq \frac{2j}{2^{k+1}} = \frac{j}{2^k} = \varphi_k(x)$$ (3)

<u>Step 2</u> $\quad \varphi_k \to f$

<u>Case 1</u> $\quad \underbrace{f(x) = +\infty}_{x \in \bigcap_{k=1}^{\infty} F_k}$

$$\Rightarrow \quad \varphi_k(x) = 2^k, \quad k = 1, 2, \ldots$$

$$\Rightarrow \quad \varphi_k(x) \to +\infty$$

<u>Case 2</u> $\quad f(x) < +\infty$

$\exists\, k_0$ s.t. $f(x) < 2^{k_0}$

$\Rightarrow \quad \forall k > k_0, \ \exists j$ s.t. $x \in E_{k,j}$

$(\because x \notin F_k)$

$$\Rightarrow \quad 0 \leq f(x) - \varphi_k(x) \leq \frac{1}{2^k}$$

(by defn of $E_{k,j}$).

$$\Rightarrow \quad \varphi_k(x) \to f(x) \quad \text{as } k \to \infty$$

如果 f 有界，同上可知：$\exists k_0$, s.t. $\forall k > k_0$

$$0 \le f(x) - \varphi_k(x) \le \frac{1}{2^k}, \quad \forall x \in \mathbb{R}^n$$

$\Rightarrow \quad \varphi_k \rightrightarrows f$

Def

$\mathrm{supp}(f) \overset{\text{def}}{=} \overline{\{f \neq 0\}}$ 称为 f 的支集

如果 $\mathrm{supp}(f)$ 是紧集，则称 f 有紧支集

Cor f 非负可测 $\Rightarrow$ $\exists$ 紧支、非负简单 $\varphi_k \nearrow f$

Pf $\exists$ 非负、简单 $\tilde{\varphi}_k \nearrow f$

令

$$\varphi_k \overset{\text{def}}{=} \tilde{\varphi}_k \cdot \chi_{B_k(0)}$$

$\Rightarrow \varphi_k$ 简单，且 $\mathrm{supp}(\varphi_k) \subset \overline{B_k(0)}$

$\forall x \in \mathbb{R}^n$，$\exists k_0$ s.t. $x \in B_{k_0}(0)$

$\Rightarrow \forall k \geq k_0 \, , \quad x \in B_k(0)$ (5)

$\Rightarrow \varphi_k(x) = \tilde{\varphi}_k(x)$

$\Rightarrow \varphi_k \nearrow f$

<u>Thm</u> $\forall$ 可測函數 f, $\exists$ 簡單函數 $\{\varphi_k\}_{k=1}^{\infty}$

s.t. $\forall x \in \mathbb{R}^n$

$$\begin{cases} 0 \leq |\varphi_1(x)| \leq |\varphi_2(x)| \leq \cdots \leq |f(x)| \\ \lim\limits_{k \to \infty} \varphi_k(x) = f(x) \end{cases}$$

<u>Pf</u> $f = f^+ - f^-$

$\exists \; \varphi_k^{(1)} \nearrow f^+ , \quad \varphi_k^{(2)} \nearrow f^-$

令 $\varphi_k \overset{\text{def}}{=} \varphi_k^{(1)} - \varphi_k^{(2)}$

$\Rightarrow \varphi_k \to f$ pointwise

<u>且</u> $|\varphi_k| = \varphi_k^{(1)} + \varphi_k^{(2)} \nearrow f^+ + f^- = |f|$

⑥

Thm 2. $\forall$ 可测函数 f，$\exists$ 阶梯函数 $\{\psi_k\}_{k=1}^{\infty}$

s.t.
$$\psi_k \to f \quad a.e.$$

Def 对集合列 $\{A_k\}_{k=1}^{\infty}$，令

$$\limsup_{k\to\infty} A_k \overset{def}{=} \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j$$

$$\liminf_{k\to\infty} A_k \overset{def}{=} \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} A_j$$

分别称为 $\{A_k\}_{k=1}^{\infty}$ 的上限集和下限集.

Rmk:

$$\limsup_{k\to\infty} A_k = \{x : \forall k,\ \exists j \geq k \ \text{s.t.}\ x \in A_j\}$$

$$\liminf_{k\to\infty} A_k = \{x : \exists k_0 \ \text{s.t.}\ \forall j \geq k_0,\ x \in A_j\}$$

Lem 1

$$\{f_k \nrightarrow f\} = \bigcup_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \left\{ |f_k - f| \geq \frac{1}{l} \right\}$$

Pf 记

$$E_{k,l} \overset{\text{def}}{=} \left\{ |f_k - f| \geq \frac{1}{l} \right\}$$

$$x \in \{ f_k \not\to f \} \iff \exists\, l \text{ s.t. 有无穷多个 } k \quad x \in E_{k,l}$$

$$\iff \exists\, l \text{ s.t. } x \in \limsup_{k\to\infty} E_{k,l}.$$

Lem 2 对可测函数列 $\{f_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$

$$\left.\begin{array}{l} f_k \to f \quad \text{a.e.} \\ \displaystyle\sum_{k=1}^{\infty} m(\{f_k \neq g_k\}) < \infty \end{array}\right\} \Rightarrow g_k \to f \quad \text{a.e.}$$

Pf $\forall \varepsilon > 0$

$$\{ |g_k - f| \geq \varepsilon \}$$

$$\subset \left\{ |f_k - g_k| \geq \frac{\varepsilon}{2} \right\} \cup \left\{ |f_k - f| \geq \frac{\varepsilon}{2} \right\}$$

$$\subset \{ f_k \neq g_k \} \cup \left\{ |f_k - f| \geq \frac{\varepsilon}{2} \right\}$$

Lem 1

$$\Longrightarrow \{g_k \not\to f\} = \bigcup_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{|g_k - f| \geq \tfrac{1}{l}\}$$

$$\subset \bigcup_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \left[\{f_k \neq g_k\} \cup \{|f_k - f| \geq \tfrac{1}{l}\} \right]$$

$$\subset \left(\limsup_{k\to\infty} \{g_k \neq f_k\} \right) \cup \{f_k \not\to f\}$$

$f_k \to f$ a.e. $\Longleftrightarrow$ $m(\{f_k \not\to f\}) = 0$

只需证:

$$m\left(\limsup_{k\to\infty} \{g_k \neq f_k\}\right) = 0$$

而

$$\sum_{k=1}^{\infty} m(\{g_k \neq f_k\}) < \infty$$

Borel-Cantelli:
$\Longrightarrow$ (Ex. 16)

$$m\left(\limsup_{k\to\infty} \{g_k \neq f_k\}\right) = 0.$$

Pf of Thm 2

Step 1 先假设 $f = \chi_E$ with $m(E) < \infty$