# 第五讲 (2026.3.16)

回忆:

外测度 $m_*: 2^{\mathbb{R}^n} \to [0, +\infty]$

$$m_*(E) \overset{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| : E \subset \bigcup_{k=1}^{\infty} Q_k \right\}$$

对 $E \subset \mathbb{R}^n$, 如果 $\forall \varepsilon > 0$, $\exists G$ 开 s.t.

$$\begin{cases} E \subset G \\ m_*(G \setminus E) < \varepsilon \end{cases}$$

则称 $E$ 是 Lebesgue 可测的.

$\mathcal{L} \overset{\text{def}}{=} \mathbb{R}^n$ 中 Lebesgue 可测集全体.

$m \overset{\text{def}}{=} m_*|_{\mathcal{L}}$ 称为 Lebesgue 测度.

Thm: $\mathcal{L}$ 是 $\mathbb{R}^n$ 上的 $\sigma$-代数

Prop 9. $\mathcal{L}$ 对可数并封闭, i.e.

$$E_k \in \mathcal{L},\ k = 1, 2, \cdots \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{L}$$

Prop 11. $\mathcal{L}$ 对差补运算封闭, i.e.

$$E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$$

Pf $\forall k$. $\exists G_k$ 开 s.t.

$$\begin{cases} E \subset G_k \\ m_*(G_k \setminus E) < \frac{1}{k} \end{cases}$$

$G_k^c$ 闭 $\Rightarrow$ $G_k^c \in \mathcal{L}$ (by Prop 10)

$\Rightarrow S \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} G_k^c \in \mathcal{L}$ (by prop 9)

$E \subset \bigcap_{k=1}^{\infty} G_k \Rightarrow S \subset E^c$

$E^c \setminus S \subset G_k \setminus E$

$\left( E^c \cap S^c = E^c \cap \left(\bigcap_{k=1}^{\infty} G_k\right) \subset E^c \cap G_k \right)$

$\Rightarrow m_*(E^c \setminus S) \leq m_*(G_k \setminus E) < \frac{1}{k}$

$\Rightarrow m_*(E^c \setminus S) = 0$

$\Rightarrow E^c \setminus S \in \mathcal{L}$ (零测集可测. prop 8)

$\Rightarrow E^c = (E^c \setminus S) \cup S \in \mathcal{L}$

Cor $\mathcal{L}$ 对可数交封闭

Def 设 $E \subset \mathbb{R}^n$，如果

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c), \quad \forall A \subset \mathbb{R}^n$$

则称 $E$ 卡氏可测

Caratheodory

Thm Lebesgue 可测 $\Longleftrightarrow$ 卡氏可测

$m \overset{def}{=} m_* \big|_{\mathcal{L}}$ 称为 Lebesgue 测度

Thm（可数可加性）

$\forall E_k \in \mathcal{L}$，$k = 1, 2, \cdots$ 互不相交，

$$m\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

Pf Step 1 先假设：$\forall k$，$E_k$ 有界

只需证：LHS $\geq$ RHS

$\forall \varepsilon > 0$，$\forall k$，$\exists$ 紧集 $F_k \subset E_k$ s.t.

$$m(E_k \setminus F_k) < \frac{\varepsilon}{2^k} \quad (\text{by Ex. 25})$$

$\forall N$，$F_1, \cdots, F_N$ 是互不相交的紧集

$\Rightarrow \quad dist(F_j, F_k) > 0, \quad \forall j \neq k.$

$$\overset{\text{Prop 4}}{\Longrightarrow} \quad m\left(\bigsqcup_{k=1}^{N} F_k\right) = \sum_{k=1}^{N} m(F_k)$$

$$\Rightarrow \quad m\left(\bigsqcup_{k=1}^{\infty} E_k\right) \geq m\left(\bigsqcup_{k=1}^{N} F_k\right)$$

$$= \sum_{k=1}^{N} m(F_k)$$

$$\geq \sum_{k=1}^{N} \left[m(E_k) - m(E_k \setminus F_k)\right]$$

$$\geq \sum_{k=1}^{N} \left[m(E_k) - \frac{\varepsilon}{2^k}\right]$$

$$\geq \sum_{k=1}^{N} m(E_k) - \varepsilon$$

$$\overset{N \to \infty}{\Longrightarrow} \quad m\left(\bigsqcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k) - \varepsilon$$

$$\Rightarrow \quad m\left(\bigsqcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k)$$

Step 2　一般情形

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} Q_k, \qquad Q_k \overset{\text{def}}{=} [-k, k]^n$$

$$\begin{cases} S_1 \overset{\text{def}}{=} Q_1 \\ S_k \overset{\text{def}}{=} Q_k \setminus Q_{k-1}, \quad k \geq 2. \end{cases}$$

$$\Rightarrow \quad \mathbb{R}^n = \bigsqcup_{k=1}^{\infty} S_k$$

$$E_{j,k} \overset{\text{def}}{=} S_j \cap E_k$$

$$\Rightarrow \quad E_k = \bigsqcup_{j=1}^{\infty} E_{j,k}$$

$$\Rightarrow \quad \bigsqcup_{k=1}^{\infty} E_k = \bigsqcup_{j,k} E_{j,k}$$

Step 1

$$\Rightarrow \quad m\Big(\bigsqcup_{k=1}^{\infty} E_k\Big) = \sum_{j,k} m(E_{j,k})$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} m(E_{j,k})$$

$$= \sum_{k=1}^{\infty} m(E_k)$$

**Def**

$$E_k \nearrow E \overset{\text{def}}{\iff} \begin{cases} E_1 \subset E_2 \subset \cdots \\ E = \bigcup_{k=1}^{\infty} E_k \end{cases}$$

$$E_k \searrow E \overset{\text{def}}{\iff} \begin{cases} E_1 \supset E_2 \supset \cdots \\ E = \bigcap_{k=1}^{\infty} E_k \end{cases}$$

Thm（测度的连续性） ⑥

(i)（向上连续）

$$E_k \nearrow E \Rightarrow m(E) = \lim_{k\to\infty} m(E_k)$$

(ii)（向下连续）

$$\left.\begin{array}{l} E_k \searrow E \\ \exists k_0 \ \text{s.t.}\ m(E_{k_0}) < \infty \end{array}\right\} \Rightarrow m(E) = \lim_{k\to\infty} m(E_k)$$

Rmk: "$\exists k_0$ s.t. $m(E_{k_0}) < \infty$" 不可去

反例: $E_k = (k, +\infty)$, $k = 1, 2, \dots$

$E_k \searrow \phi$

$\forall k$, $m(E_k) = +\infty$, 而 $m(\phi) = 0$

Pf (i) 令

$$\begin{cases} \tilde{E}_1 \overset{def}{=} E_1 \\ \tilde{E}_k \overset{def}{=} E_k \setminus E_{k-1}, & k \geq 2 \end{cases}$$

$\Rightarrow \tilde{E}_k \in \mathcal{L}$, $k = 1, 2, \dots$ s.t.

$$E = \bigsqcup_{k=1}^{\infty} \tilde{E}_k$$

$$\Rightarrow \quad m(E) = \sum_{k=1}^{\infty} m(\tilde{E}_k)$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} m(\tilde{E}_k)$$

$$= \lim_{N\to\infty} m\left(\bigsqcup_{k=1}^{N} \tilde{E}_k\right)$$

$$= \lim_{N\to\infty} m(E_N)$$

(ii) 不妨设 $m(E_1) < \infty$

令 $\tilde{E}_k = E_k \setminus E_{k+1}$, $k = 1, 2, \cdots$

$$\Rightarrow \quad E_1 = E \sqcup \left(\bigsqcup_{k=1}^{\infty} \tilde{E}_k\right)$$

$$\left(\text{Note: } E_1 \setminus E = \bigcup_{k=1}^{\infty} (E_k \setminus E_{k+1})\right)$$

$$\Rightarrow \quad m(E_1) = m(E) + \sum_{k=1}^{\infty} m(\tilde{E}_k)$$

$$= m(E) + \lim_{N\to\infty} \sum_{k=1}^{N-1} [m(E_k) - m(E_{k+1})]$$

$$= m(E) + m(E_1) - \lim_{N\to\infty} m(E_N)$$

$$\Rightarrow \quad m(E) = \lim_{N\to\infty} m(E_N)$$

Thm 设 $E \in \mathcal{L}$ ⑧

1° $\forall \varepsilon > 0$，$\exists G$ 开 s.t.

$$\begin{cases} E \subset G \\ m(G \setminus E) < \varepsilon \end{cases}$$

2° $\forall \varepsilon > 0$，$\exists F$ 闭 s.t.

$$\begin{cases} F \subset E \\ m(E \setminus F) < \varepsilon \end{cases}$$

3° 如果 $m(E) < \infty$，则：$\forall \varepsilon > 0$，

$\exists K$ 紧 s.t.

$$\begin{cases} K \subset E \\ m(E \setminus K) < \varepsilon \end{cases}$$

4° 如果 $m(E) < \infty$，则 $\forall \varepsilon > 0$，

$\exists Q_1, \cdots, Q_N$ s.t.

$$m\left(E \,\Delta \left(\bigcup_{k=1}^{N} Q_k\right)\right) < \varepsilon$$

这里

$$E_1 \Delta E_2 \overset{def}{=} (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$$

称为 $E_1$ 和 $E_2$ 的对称差.

(7)

Pf 1° 由可测的定义

2° $E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$

$\forall \varepsilon > 0$, $\exists G$ 开 s.t.

$$\begin{cases} E^c \subset G \\ m(G \setminus E^c) < \varepsilon \end{cases}$$

令 $F \stackrel{def}{=} G^c$ 闭

$$\Rightarrow \begin{cases} F \subset E \\ E \setminus F = G \setminus E^c \end{cases}$$

$$\Rightarrow m(E \setminus F) = m(G \setminus E^c) < \varepsilon$$

3° 令 $Q_k \stackrel{def}{=} [-k, k]^n$, $k = 1, 2, \cdots$

$$\Rightarrow E \cap Q_k \nearrow E$$

测度的连续性 $\Rightarrow$ $\forall \varepsilon > 0$, $\exists k$ s.t.

$$m(E \cap Q_k) > m(E) - \frac{\varepsilon}{2}$$

另一方面，由 2°，$\exists K$ 闭 s.t.

$$\begin{cases} K \subset E \cap Q_k & (\Rightarrow K \text{ 紧}) \\ m((E \cap Q_k) \setminus K) < \varepsilon/2 \end{cases}$$

$\Rightarrow m(E \setminus K)$

$$\leq m(E \setminus (E \cap Q_k)) + m((E \cap Q_k) \setminus K)$$

$$< \varepsilon$$

4° 由 $m_*$ 的定义

$\forall \varepsilon > 0$, $\exists \{Q_k\}_{k=1}^{\infty}$ s.t.

$$\begin{cases} E \subset \bigcup_{k=1}^{\infty} Q_k \\ \sum_{k=1}^{\infty} |Q_k| < m(E) + \varepsilon/2 \end{cases}$$

$\Rightarrow \exists N$ s.t.

$$\sum_{k=N+1}^{\infty} |Q_k| < \varepsilon/2$$

令 $F = \bigcup_{k=1}^{N} Q_k$

$$\Rightarrow E \setminus F \subset \left(\bigcup_{k=1}^{\infty} Q_k\right) \setminus F \subset \bigcup_{k=N+1}^{\infty} Q_k$$

$$\Rightarrow m(E \setminus F) \leq m\left(\bigcup_{k=N+1}^{\infty} Q_k\right)$$

$$\leq \sum_{k=N+1}^{\infty} |Q_k| < \varepsilon/2$$

而

$$m(F \setminus E) \leq m\left(\left(\bigcup_{k=1}^{\infty} Q_k\right) \setminus E\right)$$

$$\leq \sum_{k=1}^{\infty} |Q_k| - m(E) < \varepsilon/2$$

Thm (Lebesgue可测集与Borel集的关系)

$E \in \mathcal{L} \iff \exists\ G_\delta$集 $G$，$\exists$ 零测集 $N_1$, s.t.

$$E = G \setminus N_1$$

$\iff \exists\ F_\sigma$集 $F$，$\exists$ 零测集 $N_2$, s.t.

$$E = F \cup N_2$$

(HW)

Pf "$\Leftarrow$" 平凡

"$\Rightarrow$"

$\forall k$, $\exists\ G_k$ 开 s.t.

$$\begin{cases} E \subset G_k \\ m(G_k \setminus E) < \dfrac{1}{k} \end{cases}$$