# 第四讲 (2026.3.11)

外测度 $m_*: 2^{\mathbb{R}^n} \to [0, +\infty]$

$$m_*(E) \stackrel{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| : \ E \subset \bigcup_{k=1}^{\infty} Q_k \right\}$$

Prop. 1 (单调性) $E_1 \subset E_2 \Rightarrow m_*(E_1) \le m_*(E_2)$

Prop. 2 (次可加性)

$$m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m_*(E_k)$$

Pf 不妨设: $\forall k$, $m_*(E_k) < \infty$.

(约定: 如果 RHS 和式中有一项为 $\infty$, 不等式平凡

$\forall \varepsilon > 0$, $\forall k$, $\exists \{Q_j^{(k)}\}_{j=1}^{\infty}$ s.t.

$$\begin{cases} E_k \subset \bigcup_{j=1}^{\infty} Q_j^{(k)} \\ \sum_{j=1}^{\infty} |Q_j^{(k)}| < m_*(E_j) + \frac{\varepsilon}{2^k} \end{cases}$$

$$\Rightarrow \ m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |Q_j^{(k)}|$$

$$\le \sum_{k=1}^{\infty} \left( m_*(E_k) + \frac{\varepsilon}{2^k} \right)$$

$$= \sum_{k=1}^{\infty} m_*(E_k) + \varepsilon$$

$$\overset{\varepsilon \to 0^+}{\Longrightarrow} \quad m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m_*(E_k)$$



**Prop 3** （外正则性）

$$m_*(E) = \inf\{ m_*(G) : G \text{ 开}, \ E \subset G \}$$

**Pf** $\forall \varepsilon > 0$, $\exists \{Q_k\}_{k=1}^{\infty}$ s.t.

$$\begin{cases} E \subset \bigcup_{k=1}^{\infty} Q_k \\ \sum_{k=1}^{\infty} |Q_k| < m_*(E) + \frac{\varepsilon}{2} \end{cases}$$

$\forall k$, $\exists P_k$ 开方体 s.t.

$$\begin{cases} Q_k \subset P_k \\ |P_k| \leq |Q_k| + \frac{\varepsilon}{2^{k+1}} \end{cases}$$

令 $G \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} P_k$

$\Longrightarrow$ $E \subset G$ 且

$$m_*(E) \leq m_*(G) \leq \sum_{k=1}^{\infty} |P_k| \leq \sum_{k=1}^{\infty} |Q_k| + \frac{\varepsilon}{2}$$

$$\leq m_*(E) + \varepsilon$$

<u>Prop 4</u>

$$\mathrm{dist}(E_1, E_2) > 0 \implies m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$$

<u>Rmk</u>: $\exists E_1, E_2 \subset \mathbb{R}^n$ s.t. $E_1 \cap E_2 = \phi$, <u>但</u>

$$m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2)$$

(Ex. 33)

<u>Pf</u> 只需证：LHS $\geq$ RHS

$\forall \varepsilon > 0$, $\exists \{Q_k\}_{k=1}^{\infty}$ s.t.

$$\begin{cases} E_1 \cup E_2 \subset \bigcup_{k=1}^{\infty} Q_k \\ \sum_{k=1}^{\infty} |Q_k| < m_*(E_1 \cup E_2) + \varepsilon \end{cases}$$

不妨设：$\forall k$,

$$\mathrm{diam}(Q_k) < \frac{1}{2} \mathrm{dist}(E_1, E_2)$$

(否则细分 $Q_k$ $\longrightarrow$ 新方体覆盖 $\{\tilde{Q}_k\}_{k=1}^{\infty}$

满足要求 <u>且</u> $\sum_{k=1}^{\infty} |\tilde{Q}_k| = \sum_{k=1}^{\infty} |Q_k|$

(由方体体积的有限可加性)

$\Rightarrow$ 每个 $Q_k$ 仅与 $E_1, E_2$ 之一相交

令

$$\Lambda_1 \overset{def}{=} \{k \in \mathbb{N} : \quad Q_k \cap E_1 \neq \phi\}$$

$$\Lambda_2 \overset{def}{=} \{k \in \mathbb{N} : \quad Q_k \cap E_2 \neq \phi\}$$

$\Rightarrow \quad \mathbb{N} = \Lambda_1 \sqcup \Lambda_2$ 且

$$E_1 \subset \bigcup_{k \in \Lambda_1} Q_k \quad , \quad E_2 \subset \bigcup_{k \in \Lambda_2} Q_k$$

$\Rightarrow \quad m_*(E_1) + m_*(E_2)$

$$\leq \sum_{k \in \Lambda_1} |Q_k| + \sum_{k \in \Lambda_2} |Q_k|$$

$$= \sum_{k=1}^{\infty} |Q_k| \leq m_*(E_1 \cup E_2) + \varepsilon$$

$\Rightarrow \quad m_*(E_1) + m_*(E_2) \leq m_*(E_1 \cup E_2)$

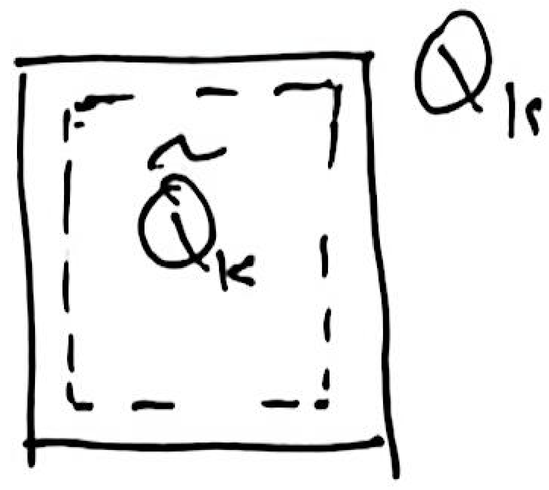
<u>Prop 5</u> 设 $\{Q_k\}_{k=1}^{\infty}$ 不重叠 (non-overlapping = 内部不交)

$$m_*\left(\biguplus_{k=1}^{\infty} Q_k\right) = \sum_{k=1}^{\infty} |Q_k| .$$

<u>Pf</u> 只需证: LHS $\geq$ RHS

$\forall \varepsilon > 0$, $\exists \{\tilde{Q}_k\}_{k=1}^{\infty}$ s.t.

$$\begin{cases} \tilde{Q}_k \subset Q_k \\ |\tilde{Q}_k| > |Q_k| - \dfrac{\varepsilon}{2^k} \\ \operatorname{dist}(\tilde{Q}_k, \tilde{Q}_j) > 0, \quad \forall k \neq j \end{cases}$$




$\Rightarrow \forall N,$

$$m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \geq m_*\left(\bigcup_{k=1}^{N} \tilde{Q}_k\right)$$

$$= \sum_{k=1}^{N} |\tilde{Q}_k| \quad \text{(by Prop 4)}$$

$$\geq \sum_{k=1}^{N} \left(|Q_k| - \frac{\varepsilon}{2^k}\right)$$

$$\overset{N\to\infty}{\Longrightarrow} m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \geq \sum_{k=1}^{\infty} |Q_k| - \varepsilon$$

$$\overset{\varepsilon\to 0^+}{\Longrightarrow} m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \geq \sum_{k=1}^{\infty} |Q_k|.$$

<u>Prop 6</u> (平移不变性)

$\forall E \subset \mathbb{R}^n, \ \forall h \in \mathbb{R}^n$

$$m_*(E+h) = m_*(E).$$

回忆 Borel $\sigma$-代数 $\mathcal{B}$

$m_*|_{\mathcal{B}}$ 称为 Borel 测度

**Def** 设 $E \subset \mathbb{R}^n$. 如果 $\forall \varepsilon > 0$, $\exists G$ 开 s.t.

$$\begin{cases} E \subset G \\ m_*(G \setminus E) < \varepsilon \end{cases}$$

则称 $E$ 是 Lebesgue 可测的 (简称可测)

$\mathcal{L} \overset{\text{def}}{=} \mathbb{R}^n$ 中 Lebesgue 可测集全体

称为 Lebesgue $\sigma$-代数.

$m \overset{\text{def}}{=} m_*|_{\mathcal{L}}$ 称为 Lebesgue 测度

**Prop 7** 开集可测

**Prop 8** 零测集可测 i.e.

$$m_*(E) = 0 \implies E \in \mathcal{L}$$

**Pf** $m_*(E) = \inf\{m_*(G) : G \text{ 开}, E \subset G\}$

$\implies \forall \varepsilon > 0$, $\exists G$ 开 s.t.

$$\begin{cases} E \subset G \\ m_*(G) < \underbrace{m_*(E)}_{=0} + \varepsilon = \varepsilon \end{cases}$$

$$\implies m_*(G \setminus E) \leqslant m_*(G) < \varepsilon$$

**Thm** $\mathcal{L}$ 是 $\mathbb{R}^n$ 上的 $\sigma$-代数.

**Prop 9** $E_k \in \mathcal{L},\ k=1,2,\cdots \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{L}$

Pf $\forall \varepsilon > 0$, $\exists\, G_k$ 开 s.t.

$$\begin{cases} E_k \subset G_k \\ m_*(G_k \setminus E_k) < \dfrac{\varepsilon}{2^k} \end{cases}$$

令 $G \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} G_k$

$$\Rightarrow G \setminus \left(\bigcup_{k=1}^{\infty} E_k\right) \subset \bigcup_{k=1}^{\infty} (G_k \setminus E_k)$$

$$\Rightarrow m_*\left(G \setminus \bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m_*(G_k \setminus E_k) < \varepsilon$$

**Prop 10** 闭集可测

Pf Step 1 紧集可测

设 $F \overset{\text{cpt}}{\subset} \mathbb{R}^n$

$\Rightarrow m_*(F) < \infty$ $(\because \exists Q$ s.t. $F \subset Q$

而 $m_*(F) = \inf\{m_*(G) : G$ 开, $F \subset G\}$

(外正则性)

$\Rightarrow \forall \varepsilon > 0, \ \exists G$ 开 s.t.

$$\begin{cases} F \subset G \\ m_*(G) < m_*(F) + \varepsilon \end{cases}$$

而 $G \setminus F$ 开

$$\xRightarrow{\text{开集结构}} \quad G \setminus F = \biguplus_{k=1}^{\infty} Q_k$$

令

$$K_N \overset{def}{=} \bigcup_{k=1}^{N} Q_k, \qquad N = 1, 2, \cdots$$

$\Rightarrow K_N$ 紧 且 $K_N \subset G \setminus F$.

$\Rightarrow \operatorname{dist}(K_N, F) > 0$

$\Rightarrow m_*(K_N \cup F) = m_*(K_N) + m_*(F)$

$$\Rightarrow m_*(K_N) = m_*(K_N \cup F) - m_*(F)$$
$$\le m_*(G) - m_*(F) < \varepsilon$$

$$\xRightarrow{N \to \infty} \quad m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \le \varepsilon \quad \text{即} \quad m_*(G \setminus F) \le \varepsilon$$

Step 2 一般情形

$$F = \bigcup_{k=1}^{\infty} (\underbrace{F \cap \overline{B_k(0)}}_{\text{紧}}) \in \mathcal{L}$$