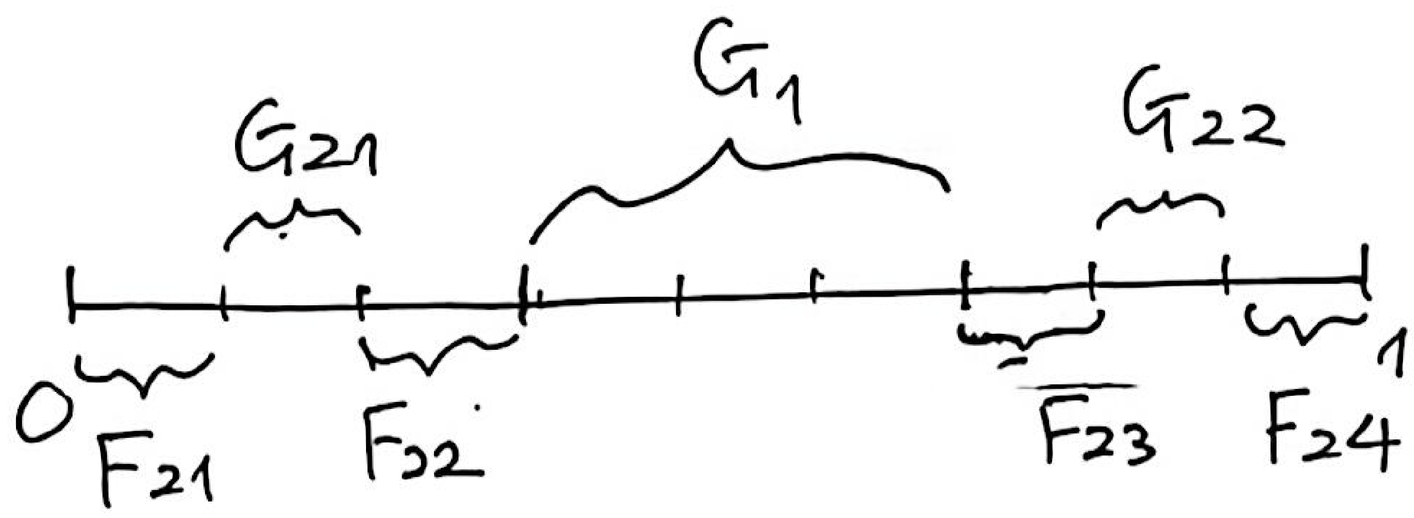
第三讲 (2026.3.9)

Cantor 集



将 $[0,1]$ 三等分，去掉居中的开区间

$$G_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$$

留下两个闭区间

$$F_{11} = \left[0, \frac{1}{3}\right], \qquad F_{12} = \left[\frac{2}{3}, 1\right]$$

记

$$F_1 \overset{\text{def}}{=} F_{11} \cup F_{12}$$

将 F_{11} 和 F_{12} 分别三等分，各去掉居中的开区间

$$G_{21} = \left(\frac{1}{3^2}, \frac{2}{3^2}\right), \qquad G_{22} = \left(\frac{7}{3^2}, \frac{8}{3^2}\right)$$

$$\vdots$$

第 n 次操作：去掉开区间

$$G_{n1} = \left(\frac{1}{3^n}, \frac{2}{3^n}\right), \quad G_{n2} = \left(\frac{7}{3^n}, \frac{8}{3^n}\right), \quad \cdots, \quad G_{n,2^{n-1}} = \left(\frac{3^n-2}{3^n}, \frac{3^n-1}{3^n}\right)$$

第n次操作：去掉2^{n-1}个开区间

$G_{n1}=\left(\frac{1}{3^n},\frac{2}{3^n}\right)$，$G_{n2}=\left(\frac{7}{3^n},\frac{8}{3^n}\right)$，…，$G_{n,2^{n-1}}=\left(\frac{3^n-2}{3^n},\frac{3^n-1}{3^n}\right)$

令

$$G_n \overset{\text{def}}{=} \bigcup_{k=1}^{2^n-1} G_{nk} \qquad \text{（开集）}$$

留下2^n个闭区间

$F_{n1}=\left[0,\frac{1}{3^n}\right]$，$F_{n2}=\left[\frac{2}{3^n},\frac{3}{3^n}\right]$，…，$F_{n,2^n}=\left[\frac{3^n-1}{3^n},1\right]$

令

$$F_n \overset{\text{def}}{=} \bigcup_{k=1}^{2^n} F_{nk} \qquad \text{（闭集）}$$

$$G \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} G_n \quad \text{称为 Cantor 开集}$$

$$C \overset{\text{def}}{=} \bigcap_{n=1}^{\infty} F_n \quad \text{称为 Cantor 三分集}$$

1° $C=[0,1]\setminus G$ 是闭集

2° $C\neq\phi$ （由闭集套定理）

3° $C=C'$ （导集，聚点全体）

即 C 无孤立点，是完全集（HW: Ex. 1）

4° $\mathrm{int}(C)=\phi$，即 C 无内点.

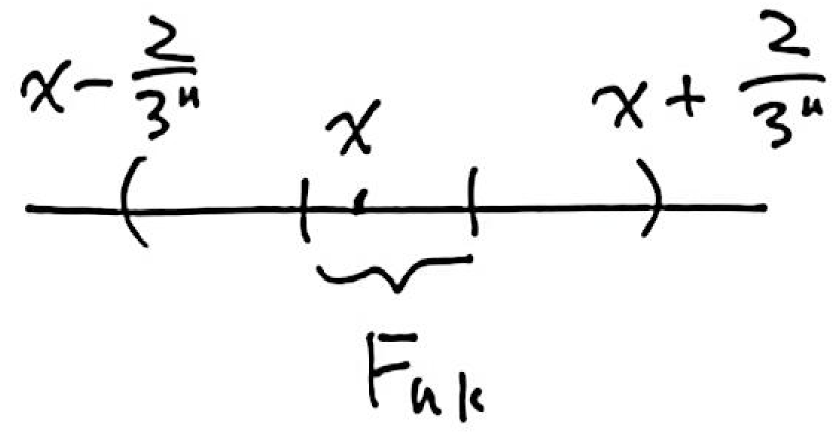
假设 $\exists x\in C$, $\exists \varepsilon>0$ s.t.

$$(x-\varepsilon,\ x+\varepsilon)\subset C$$

取 n 充分大 s.t. $\frac{2}{3^n}<\varepsilon$.

$x\in C \Rightarrow \exists F_{nk}$ s.t. $x\in F_{nk}$

而 $|F_{nk}|=\frac{1}{3^n}$



$$\Rightarrow F_{nk}\subset\left(x-\frac{2}{3^n},\ x+\frac{2}{3^n}\right)$$

$$\subset (x-\varepsilon,\ x+\varepsilon)\subset C$$

而在 Cantor 集构造过程的第 $n+1$ 步要去掉 $G_{n+1,k}\subset F_{nk}$，故 $(x-\varepsilon,\ x+\varepsilon)$ 中一定含有 $[0,1]\setminus C$ 中的点. 矛盾.

5° C 具有连续统基数 i.e. 存在 $[0,1]$ 到 C 的一一映射 (HW. Ex. 2)

6° Cantor 开集 G 中开区间长度之和 $=1$.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}|G_{nk}|=\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n=1.$$

$\Rightarrow$ Cantor 集是零集.

7° $\underbrace{\dim_H}_{\text{Hausdorff 维数}} C = \dfrac{\log 2}{\log 3} = 0.6309\cdots$

外测度.

对长方体 $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$

定义其体积

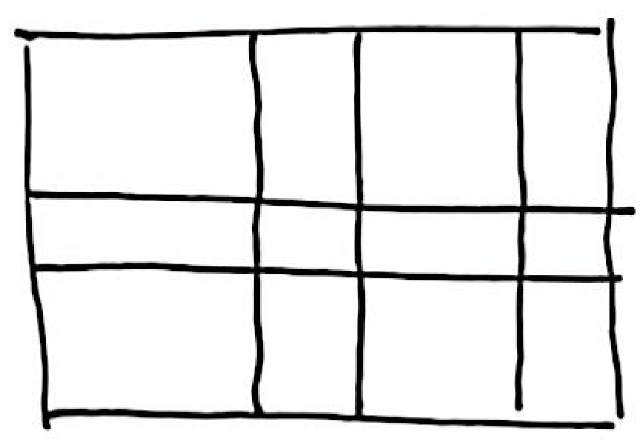
$$|R| \overset{\text{def}}{=} \prod_{k=1}^{n} (b_k - a_k)$$

$\biguplus$ — 内部不交并

<u>Lem 1</u> 如果 $R = \biguplus_{k=1}^{N} R_k$，$R, R_k$ 均为长方体，则

$$|R| = \sum_{k=1}^{N} |R_k|$$

Pf <u>Case 1</u>. $R_1, \dots, R_N$ 构成 R 的网格状覆盖



$n=1$ 时

$$[a, b] = [a_1, b_1] \uplus \cdots \uplus [a_N, b_N]$$

$$a = a_1 < b_1 = a_2 < \cdots < b_{N-1} = a_N < b_N = b$$

平凡

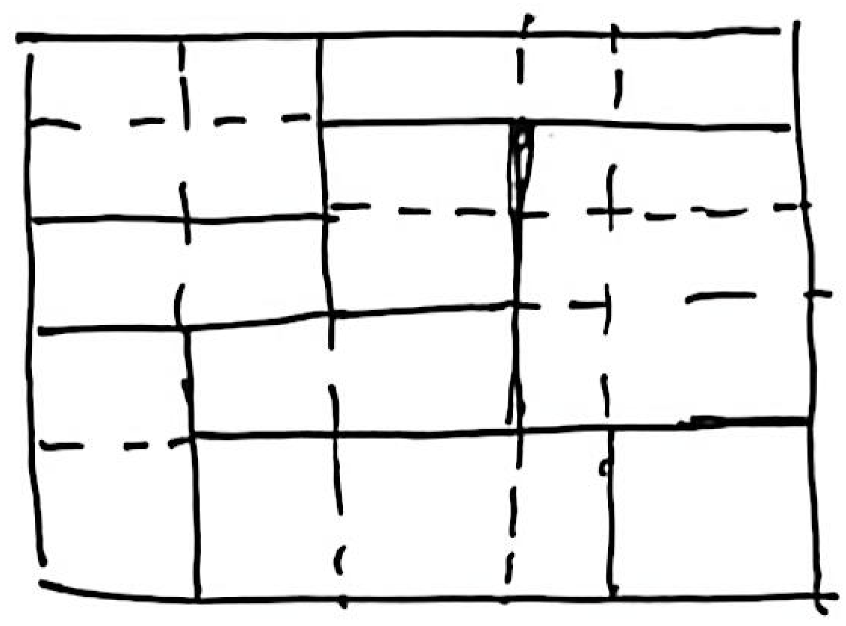
$n=2$ 时 $\quad R = I \times J$

$$I = I_1 \uplus \cdots \uplus I_m, \qquad J = J_1 \uplus \cdots \uplus J_\ell$$

$$\begin{aligned}
|R| &= |I| \times |J| \\
&= \Big(\sum_{j=1}^{m} |I_j|\Big)\Big(\sum_{k=1}^{\ell} |J_k|\Big) \\
&= \sum_{j=1}^{m} \sum_{k=1}^{\ell} |I_j|\,|J_k| \\
&= \sum_{j=1}^{m} \sum_{k=1}^{\ell} |I_j \times J_k| = \sum_{j=1}^{m} \sum_{k=1}^{\ell} |R_{jk}|
\end{aligned}$$

$n \geq 3$ 归纳法.

<u>Case 2</u> 一般情形



延长每个 R_k 的各个面

$\Rightarrow$ R 的网格状覆盖 $\{\tilde{R}_1, \dots, \tilde{R}_m\}$

$$R_k = \biguplus_{j \in \Lambda_k} \tilde{R}_j$$

$$\{1, \dots, m\} = \Lambda_1 \sqcup \cdots \sqcup \Lambda_N$$

Case 1.

$$\Rightarrow \quad |R| = \sum_{j=1}^{m} |\tilde{R}_j| = \sum_{k=1}^{N} \sum_{j \in \Lambda_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|$$

Lem 2 $\quad R \subset \bigcup_{k=1}^{N} R_k \Rightarrow |R| \leq \sum_{k=1}^{N} |R_k|$ ⑥

(HW)

Def 如下定义的集函数 $m_*: 2^{\mathbb{R}^n} \to [0,+\infty]$

$$m_*(E) \overset{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| : \{Q_k\}_{k=1}^{\infty} \text{ s.t. } E \subset \bigcup_{k=1}^{\infty} Q_k \right\}$$

称为 E 的外测度.

例: $m_*(\phi) = 0$

$m_*(\{a\}) = 0$

例: $m_*(C) = 0$, C 是 Cantor 三分集

$$C = \bigcap_{k=1}^{\infty} F_k \quad \text{with} \quad F_k = \bigcup_{j=1}^{2^k} F_{k,j}$$

$$\Rightarrow \quad C \subset \bigcup_{j=1}^{2^k} F_{k,j}$$

$$\Rightarrow \quad m_*(C) \leq \sum_{j=1}^{2^k} |F_{k,j}| = \frac{2^k}{3^k} \to 0$$

Prop: $m_*(R) = |R|$, R 为长方体.

Pf: 1° $\quad |R| \leq m_*(R)$

$\forall \varepsilon > 0$, $\exists \{Q_k\}_{k=1}^{\infty}$ s.t.

(i) $R \subset \bigcup_{k=1}^{\infty} Q_k$

(ii) $\sum_{k=1}^{\infty} |Q_k| < m_*(R) + \varepsilon$

对每个 Q_k, $\exists \tilde{Q}_k$ s.t.

$$Q_k \subset \operatorname{int}(\tilde{Q}_k) \quad \underline{\text{W}} \quad |\tilde{Q}_k| < (1+\varepsilon)|Q_k|$$

$$\Rightarrow \quad R \subset \bigcup_{k=1}^{\infty} \operatorname{int}(\tilde{Q}_k)$$

R 紧.

$\Rightarrow$ $\exists N$ s.t.

$$R \subset \bigcup_{k=1}^{N} \operatorname{int}(\tilde{Q}_k) \subset \bigcup_{k=1}^{N} \tilde{Q}_k$$

Lem 2

$$\Rightarrow \quad |R| \leq \sum_{k=1}^{N} |\tilde{Q}_k|$$

$$\leq (1+\varepsilon) \sum_{k=1}^{N} |Q_k|$$

$$< (1+\varepsilon)(m_*(R) + \varepsilon)$$

$\varepsilon \to 0^+$

$$\Rightarrow \quad |R| \leq m_*(R)$$

2° $m_*(R) \leq |R|$

$$\Gamma_k \overset{\text{def}}{=} \left\{ 2^{-k}\left([0,1]^n + m\right) : m \in \mathbb{Z}^n \right\}$$

$$\mathcal{F}_k \overset{\text{def}}{=} \left\{ Q \in \Gamma_k : Q \cap R \neq \phi \right\}$$

$$\mathcal{F}_k' \overset{\text{def}}{=} \left\{ Q \in \mathcal{F}_k : Q \subset R \right\}$$

$$\mathcal{F}_k'' \overset{\text{def}}{=} \left\{ Q \in \mathcal{F}_k : Q \not\subset R \right\}$$

$\Longrightarrow$ $\mathcal{F}_k = \mathcal{F}_k' \sqcup \mathcal{F}_k''$ 是 R 的一个方体覆盖.

Claim $\# \mathcal{F}_k'' = O(2^{k(n-1)})$

$$\mathcal{F}_k'' \subset \left\{ Q \in \Gamma_k : Q \cap \partial R = \phi \right\}$$

$$\# \left\{ Q \in \Gamma_k : Q \cap \partial R = \phi \right\} \lesssim \frac{\text{Area}(\partial R) \times 2^{-k} \times 2}{2^{-kn}} = O(2^{k(n-1)})$$

$$\Longrightarrow \sum_{Q \in \mathcal{F}_k''} |Q| \leq C \cdot 2^{k(n-1)} \cdot 2^{-kn} = C \cdot 2^{-k}$$

而 $\displaystyle\sum_{Q \in \mathcal{F}_k'} |Q| \leq |R|$.

(9)

$$\Rightarrow \quad m_*(R) \leq \sum_{Q \in \mathcal{F}_k} |Q| \leq |R| + C \cdot 2^{-k}$$

$k \to \infty$

$$\Rightarrow \quad m_*(R) \leq |R| .$$